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BUFFER-STOCK SAVING IN A KRUSELL-SMITH WORLD

Christopher D. Carroll, Jiri Slacalek
and Kiichi Tokuoka

In 2014 all ECB publications feature a motif taken from the €20 banknote.

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Christopher D. Carroll
Johns Hopkins University; e-mail: ccarroll@jhu.edu; http://econ.jhu.edu/people/ccarroll/

Jiri Slacalek
European Central Bank; e-mail: jiri.slacalek@ecb.europa.eu; http://www.slacalek.com/

Kiichi Tokuoka
Ministry of Finance, Japan; e-mail: kiichi.tokuoka@mof.go.jp

Address	Kaiserstrasse 29, 60311 Frankfurt am Main, Germany
Postal address	Postfach 16 03 19, 60066 Frankfurt am Main, Germany
Telephone	+49 69 1344 0
Internet	http://www.ecb.europa.eu
Fax	+49 69 1344 6000

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Abstract

A large body of microeconomic evidence supports Friedman (1957)'s proposition that household income can be reasonably well described as having both transitory and permanent components. We show how to modify the widely-used macroeconomic model of Krusell and Smith (1998) to accommodate such a microeconomic income process. Our incorporation of substantial permanent income shocks helps our model to explain a substantial part of the large degree of empirical wealth heterogeneity that is unexplained in the baseline Krusell and Smith (1998) model, even without heterogeneity in preferences.

Keywords Microfoundations of Macroeconomics, Household Income Process, Aggregate Uncertainty, Wealth Inequality

JEL codes D12, D31, D91, E21

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Non-technical Summary

A large body of microeconomic evidence supports Friedman (1957)'s proposition that household income can be reasonably well described as having both transitory and permanent components. We show how to modify the widely-used macroeconomic model of Krusell and Smith (1998) (KS) to accommodate such a microeconomic income process. Our incorporation of substantial permanent income shocks helps our model to explain a substantial part of the large degree of empirical wealth heterogeneity that is unexplained in the baseline Krusell and Smith (1998) model, even without heterogeneity in preferences.

Although the Krusell and Smith (1998) method for incorporating uninsurable idiosyncratic risk into macroeconomic models has become a workhorse with a wide range of applications, its plausible criticism is that their assumed stochastic process for household income bears little relationship to microeconomic evidence about household income dynamics. Since the purpose of KS modeling exercise was to derive quantitative implications of idiosyncratic uncertainty, it is hard to be confident about their (quantitative) conclusions if the calibration of the idiosyncratic risk is (quantitatively) implausible.

We review a large body of microeconomic evidence which finds that a simple income process consisting of a permanent (random walk) and a transitory (white noise) component—what we call the "Friedman/Buffer Stock" (FBS) process—captures the key features of the microeconomic data well. We then solve a modified version of the Krusell and Smith (1998) model in which the household income process has been calibrated to be consistent with the FBS household income process. The main model modification is that it is necessary for households to have finite, rather than infinite, lifetimes. Along with a plausible assumption about the permanent income of newborns, if the exogenous risk of death *a la* Blanchard (1985) is large enough, we show that the cross-section distribution of permanent income is stable. (This overcomes the perceived obstacle to incorporation of permanent shocks that in infinite-horizon contexts, a model with permanent shocks does not have an ergodic distribution of the level of permanent income).

Our variant of the KS model with the FBS household income process is actually substantially easier to solve than the original Krusell–Smith model. It also produces results that are closer to the data in an additional dimension beyond microeconomic income dynamics: The substantial permanent component in income translates into considerable heterogeneity in wealth across households. Our simulations document that top 1 percent of households in our model are three times richer than the top 1 percent in the original baseline KS model (although both setups fall short of the degree of inequality found in the empirical data).

1 Introduction

The Krusell and Smith (1998) method for incorporating uninsurable idiosyncratic risk into macroeconomic models has become such a workhorse that the *Journal of Economic Dynamics and Control* devoted a special issue to competing algorithms for implementing the method (Journal of Economic Dynamics and Control (2010)). But a plausible criticism of the Krusell and Smith (1998) model is that their assumed stochastic process for household income bears little relationship to microeconomic evidence about household income dynamics. Since the purpose of their modeling exercise was to derive quantitative implications of idiosyncratic uncertainty, it is hard to be confident about their (quantitative) conclusions if the calibration of the idiosyncratic risk is (quantitatively) implausible.

We review a large body of microeconomic evidence which finds that a simple income process consisting of a permanent (random walk) and a transitory (white noise) component—what we call the "Friedman/Buffer Stock" (FBS) process—captures the key features of the microeconomic data well. We then solve a modified version of the Krusell and Smith (1998) model in which the household income process has been calibrated to be consistent with the FBS household income process. The main model modification is that it is necessary for households to have finite, rather than infinite, lifetimes. Along with a plausible assumption about the permanent income of newborns, if the exogenous risk of death *a la* Blanchard (1985) is large enough, we show that the cross-section distribution of permanent income is stable. (This overcomes the perceived obstacle to incorporation of permanent shocks that in infinite-horizon contexts, a model with permanent shocks does not have an ergodic distribution of the level of permanent income).

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The paper is structured as follows. Section 2.1 articulates the perfect foresight framework from which the model can be viewed as a deviation. Section 2.2 describes the FBS income process. Section 2.3 introduces the risk of death and shows how it ensures a stable distribution of income. Section 2.4 describes results from the model with idiosyncratic uncertainty driven by the FBS household income process. Section 3 calibrates the model. Section 4 presents the simulated wealth distribution and compares it to that in the KS

[1] Carroll, Slacalek, and Tokuoka (2013) show that when a modest amount of heterogeneity in impatience is added to the models, they are able to match the wealth distribution much better; one interpretation of our results is that the degree of preference heterogeneity need not be so large as Krusell and Smith (1998) proposed in order for a model to match the data.
Large literature has explored alternative strategies to match the empirical wealth distribution, such as accounting for entrepreneurial choice (Quadrini (2000)), bequests (De Nardi (2004)), a combination of credit constraints and non-convexities (Banerjee and Newman (1993)), or directly calibrating the income process (Castaneda, Diaz-Gimenez, and Rios-Rull (2003)).

model and in the data. Section 5 describes the full-blown model with the FBS household income process and KS aggregate shocks, and section 6 concludes.

2 Model

2.1 The Perfect Foresight Representative Agent Model

To establish notation and a transparent benchmark, we begin by briefly sketching a standard perfect foresight representative agent model.

The aggregate production function is

$$Z_t \boldsymbol{K}_t^{\alpha} (\ell \boldsymbol{L}_t)^{1-\alpha}, \tag{1}$$

where Z_t is aggregate productivity in period t, $\boldsymbol{K}_t$ is capital, ℓ is time worked per employee, and $\boldsymbol{L}_t$ is employment. The representative agent's goal is to maximize discounted utility from consumption

$$\max \sum_{n=0}^{\infty} \beta^n \mathrm{u}(\boldsymbol{C}_{t+n})$$

for a CRRA utility function $\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$.[2] The representative agent's state at the time of the consumption decision is defined by two variables: $\boldsymbol{M}_t$ is market resources, and Z_t is aggregate productivity.

The transition process for $\boldsymbol{M}_t$ is broken up, for clarity of analysis and consistency with later notation, into three steps. Assets at the end of the period are equal to market resources minus consumption,

$$\boldsymbol{A}_t = \boldsymbol{M}_t - \boldsymbol{C}_t,$$

while next period's capital is determined from this period's assets via

$$\boldsymbol{K}_{t+1} = \boldsymbol{A}_t.$$

The final step can be conceived as the transition from the beginning of period $t+1$ when capital has not yet been used to produce output, to the middle of that period, when output has been produced and incorporated into resources but has not yet been consumed:

$$\boldsymbol{M}_{t+1} = (1-\delta)\boldsymbol{K}_{t+1} + \underbrace{Z_{t+1}\boldsymbol{K}_{t+1}^{\alpha}(\ell \boldsymbol{L}_{t+1})^{1-\alpha}}_{\boldsymbol{K}_{t+1}\mathsf{r}_{t+1} + (\ell \boldsymbol{L}_{t+1})\mathsf{W}_{t+1}},$$

where r_{t+1} is the interest rate,[3] W_{t+1} is the wage rate,[4] and $(1-\delta)$ is the depreciation factor for capital.

[2] Substitute $\mathrm{u}(\bullet) = \log \bullet$ for the case where $\rho = 1$.

[3] Equal to the marginal product of capital, $\alpha Z_{t+1}\boldsymbol{K}_{t+1}^{\alpha-1}(\ell \boldsymbol{L}_{t+1})^{1-\alpha}$.

[4] Equal to the marginal product of labor, $(1-\alpha)Z_{t+1}\boldsymbol{K}_{t+1}^{\alpha}(\ell \boldsymbol{L}_{t+1})^{-\alpha}$.

After normalizing by effective labor supply, $\mathbf{P}_t = Z_t^{1/(1-\alpha)}(\ell \boldsymbol{L}_t)$,[5] the representative agent's problem is

$$\begin{aligned} \mathrm{V}(M_t, Z_t) &= \max_{\mathrm{C}_t} \ \mathrm{u}(\mathrm{C}_t) + \beta \, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{V}(M_{t+1}, Z_{t+1}) \right] && (2) \\ & \text{s.t.} \\ A_t &= M_t - C_t, && (3) \\ K_{t+1} &= A_t / \Gamma_{t+1}, && (4) \\ M_{t+1} &= (1-\delta) K_{t+1} + K_{t+1}^{\alpha}, && (5) \end{aligned}$$

where the non-bold variables are the corresponding bold variables divided by $Z_t^{1/(1-\alpha)}(\ell \boldsymbol{L}_t)$ (e.g., $A_t = \boldsymbol{A}_t / \mathbf{P}_t$), and the growth factor for labor's effective productive power is $\Gamma_{t+1} = \mathbf{P}_{t+1}/\mathbf{P}_t$. The expectations operator $\mathbb{E}_t$ here signifies the perfection of the agent's foresight (but will have the usual interpretation when uncertainty is introduced below).

2.2 The Household Income Process *a la* Friedman (1957)

Before proceeding to a description of the model with idiosyncratic uncertainty (in section 2.4), we discuss its two ingredients: the household income process (in this section) and finite lifetimes (in the next section).

A large empirical literature summarized in Table 1 below has over the past several decades analyzed household income dynamics. For our purposes, the principal conclusion from this literature is that household income can be reasonably well described as follows. The idiosyncratic permanent component of labor income p evolves according to

$$p_{t+1} = \Gamma p_t \psi_{t+1} \tag{6}$$

where Γ captures the predictable low-frequency (e.g., life-cycle and demographic) components of income growth, and the Greek letter psi mnemonically indicates the **p**ermanent **s**hock to **i**ncome. Actual income is the product of permanent income, a mean-one transitory shock, and the wage rate:

$$\boldsymbol{y}_{t+1} = p_{t+1} \xi_{t+1} \mathsf{W}_{t+1}.$$

After taking logarithms, this income process is strikingly similar to Friedman (1957)'s characterization of income as having permanent and transitory components. Because this process has been used widely in the literature on buffer stock saving, and though similar to Friedman's formulation is not identical to it, we henceforth refer to it as the Friedman/Buffer Stock (or 'FBS') process.[6,7]

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[5] Details of this normalization are discussed in Carroll (2000).

[6] Guvenen (2007) refers to a process like this one as a 'restricted income process' (RIP) as distinguished from a process that he proposes which is similar but which allows each individual to have a distinct idiosyncratic mean growth rate. Guvenen's argument that each household has its own growth rate is intuitively plausible (indeed, it occurred to earlier authors who tested and rejected it), but Hryshko (2012) argues that there is no evidence that the Guvenen income process describes the data better (in a quantitatively meaningful way) than the restricted income process. Since incorporation of Guvenen's income process introduces serious modeling difficulties, it seems prudent to avoid using it unless the evidence for idiosyncratic growth factors becomes compelling.

[7] Friedman (1957)'s formulation was in levels rather than logs; we call ours a "buffer stock" process to distinguish it

2.3 Finite Lifetimes and the Finite Cross-Sectional Variance of Income

One might wish to use the FBS income process specified above as a complete characterization of household income dynamics, but that idea has a problem: Since each household accumulates a permanent shock in every period, the cross-sectional distribution of idiosyncratic permanent income becomes wider and wider indefinitely as the simulation progresses; that is, there is no ergodic distribution of permanent income in the population.

This problem and several others can be addressed by assuming that the model's agents have finite lifetimes *a la* Blanchard (1985). Death follows a Poisson process, so that every agent alive at date t has an equal probability D of dying before the beginning of period $t+1$. (The probability of *not* dying is the cancelation of the probability of dying: $\not{\mathsf{D}} = 1 - \mathsf{D}$). Households engage in a Blanchardian mutual insurance scheme: Survivors share the estates of those who die. Assuming a zero profit condition for the insurance industry, the insurance scheme's ultimate effect is simply to boost the rate of return (for survivors) by an amount exactly corresponding to the mortality rate.

In order to maintain a constant population (of mass one, uniformly distributed on the unit interval), we assume that dying households are replaced by an equal number of newborns; we write the population-mean operator as $\mathbb{M}[\bullet_t] = \int_0^1 \bullet_{t,\iota} d\iota$. Newborns, we assume, begin life with a level of idiosyncratic permanent income equal to the mean level of idiosyncratic permanent income in the population as a whole. Conveniently, our definition of the permanent shock implies that in a large population, mean idiosyncratic permanent income will remain fixed at $\mathbb{M}[p] = 1$ forever, while the mean of p^2 is given by[8]

$$\mathbb{M}[p^2] = \frac{\mathsf{D}}{1 - \not{\mathsf{D}}\mathbb{E}[\psi^2]} \tag{7}$$

and the variance of p by

$$\sigma_p^2 = \mathbb{M}[p^2] - 1.$$

Of course for all of this to be valid, it is necessary to impose the parametric restriction $\not{\mathsf{D}}\mathbb{E}[\psi^2] < 1$ (a requirement that does not do violence to the data, as we shall see). Intuitively, the requirement is that, among surviving consumers, income does not spread out so quickly as to overwhelm the compression of the permanent income distribution that arises because of the equalizing force of death and replacement.

from Friedman's formulation and because it has been widely used in the literature on buffer-stock saving. Some papers, instead of imposing a random walk, have allowed for an AR(1) persistent component; but our reading of the literature is that whenever those papers have also allowed for an MA(1) transitory component—as would be implied by any framework in which transitory shocks occur on dates other than January 1—the AR(1) coefficient is always very close to 1.

[8]See Appendix A for the derivation.

2.4 Putting the (Microeconomic) Parts Together

We now introduce the FBS household income process of section 2.2 and finite lifetimes of section 2.3 into the perfect foresight model of section 2.1. (Below we extend the model to incorporate aggregate shocks *a la* Krusell and Smith (1998)).

Extending section 2.2, for convenience setting $\Gamma = 1$, the process of noncapital income of each household follows

$$\boldsymbol{y}_t = p_t \xi_t \mathsf{W}_t, \tag{8}$$

$$p_t = p_{t-1}\psi_t, \tag{9}$$

$$\mathsf{W}_t = (1-\alpha) Z_t (\boldsymbol{K}_t/\ell \boldsymbol{L}_t)^\alpha, \tag{10}$$

where $\boldsymbol{y}_t$ is noncapital income for the household in period t, equal to the permanent component of noncapital income p_t multiplied by a transitory income shock factor ξ_t and wage rate W_t; the permanent component of noncapital income in period t is equal to its previous value, multiplied by a mean-one iid shock ψ_t, $\mathbb{E}_t[\psi_{t+n}] = 1$ for all $n \geq 1$. $\boldsymbol{K}_t$ is capital and $\boldsymbol{L}_t = 1 - u_t$ is the employment rate (because u_t is the unemployment rate). Since there is no aggregate shock, Z_t, $\boldsymbol{K}_t$, $\boldsymbol{L}_t$, and W_t are constant ($Z_t = Z = 1$, $\boldsymbol{K}_t = \boldsymbol{K}$, $\boldsymbol{L}_t = \boldsymbol{L}$, and $\mathsf{W}_t = \mathsf{W} = (1-\alpha)(\boldsymbol{K}/\ell\boldsymbol{L})^\alpha$).

Following the assumptions in the the special issue of the *Journal of Economic Dynamics and Control* (Journal of Economic Dynamics and Control (2010)) devoted to comparing solution methods for the KS model, the distribution of ξ_t is:

$$\xi_t = \mu \text{ with probability } u_t, \tag{11}$$

$$= (1-\tau_t)\ell\theta_t \text{ with probability } 1-u_t, \tag{12}$$

where $\mu > 0$ is the unemployment insurance payment when unemployed and $\tau_t = \mu u_t / \ell \boldsymbol{L}_t$ is the rate of tax collected to pay unemployment benefits (see Table 3 for parameter values).[9] The probability of unemployment is constant ($u_t = u$); later we allow u to vary over time.

The decision problem for the household in period t can be written using normalized variables; the consumer's objective is to choose a series of consumption functions c between now and the end of the horizon that satisfy:

$$\mathrm{v}(m_t) = \max_{\mathrm{c}_t} \ \mathrm{u}(\mathrm{c}_t) + \beta \cancel{D} \mathbb{E}_t \left[\psi_{t+1}^{1-\rho} \mathrm{v}(m_{t+1})\right] \tag{13}$$

s.t.

$$a_t = m_t - c_t,$$

$$a_t \geq 0,$$

$$k_{t+1} = a_t/(\cancel{D}\psi_{t+1}), \tag{14}$$

$$m_{t+1} = ((1-\delta)+r)k_{t+1} + \xi_{t+1}, \tag{15}$$

where the non-bold ratio variables are defined as the bold (level) variables divided by the level of permanent income $\boldsymbol{p}_t = p_t \mathsf{W}$ (e.g., $m_t = \boldsymbol{m}_t/(p_t\mathsf{W})$). The only state variable is (normalized) cash-on-hand m_t. The household's employment status is *not* a state vari-

[9] The original KS model assumed no unemployment insurance ($\mu = 0$).

able, unlike in the KS model, where tomorrow's employment status depends on today's status. This substantially simplifies the analysis (which is useful for computational and analytical purposes), arguably without too much sacrifice of realism (except possibly for detailed studies of the behavior of households during extended unemployment spells).

Since households die with a constant probability D between periods, the effective discount factor is $\beta\cancel{\mathsf{D}}$ (in (13)); the effective interest rate is $\big((1-\delta)+r\big)/\cancel{\mathsf{D}}$ (combining (14) and (15)).[10]

3 Calibration

This section discusses the calibration of the model with a special focus on two key features: the income process and the time preference factor. The model is calibrated at the quarterly frequency.

3.1 Parametrization of the Income Process

We first calibrate the income process using the existing empirical literature and households' subjective estimates of permanent income. Table 1 summarizes the annual variances of log permanent shocks (σ^2_ψ) and log transitory shocks (σ^2_ξ) estimated by a selection of papers from the extensive literature based both on the US and international data; see also Meghir and Pistaferri (2011) for a fuller literature review and Review of Economic Dynamics (2010) for international evidence.[11] Some authors have used a process of this kind to describe the labor income or wage process for an individual worker (top panel), while others have used it to describe the process for overall household income (bottom panel); it seems to work reasonably well in both cases (though, obviously, with different estimates of the variances).[12]

The last line of the table shows what labor economists would have found, when estimating a process like the one above, if the empirical data were generated by households who experienced an income process like the one assumed by the KS-JEDC model.[13] This row of the table makes our point forcefully: The empirical procedures that have actually been applied to empirical micro data, if used to measure the income process households experience in a KS economy, would have produced estimates of σ^2_ψ and σ^2_ξ that are orders of magnitude different from what the actual empirical literature finds in actual data.

[10] The term $\big((1-\delta)+r\big)$ is scaled by $1/\cancel{\mathsf{D}}$ due to the Blanchardian mutual insurance scheme as described in the previous subsection.

[11] Most authors cited above used US data. Nielsen and Vissing-Jorgensen (2006) used Danish data and estimated $\sigma^2_\psi = 0.005$ and $\sigma^2_\xi = 0.015$. It would be reasonable to interpret their estimates as the lower bounds for the US, given that their administrative data is well-measured and but that Danish welfare is more generous than the US system.

[12] Recent work by Sabelhaus and Song (2010) using newly available data from Social Security earnings files finds that the variances of both transitory and permanent shocks have declined during the "Great Moderation" period at all ages; they also find distinct life cycle patterns of shocks by age, with young people experiencing higher levels of both kinds of shocks than the middle-aged).

[13] First, we generated income draws according to the income process in the KS-JEDC model. Then, following the method in Carroll and Samwick (1997), we estimated the variances under the assumption that these income draws were produced by the process $\boldsymbol{y}_t = p_t \xi_t$ where $p_t = p_{t-1}\psi_t$. In doing so, as in Carroll and Samwick (1997), the draws of $\boldsymbol{y}_t$ are excluded when $\boldsymbol{y}_t$ is very low relative to its mean (see Carroll and Samwick (1997) for details about this restriction).

Table 1 Estimates of Annual Variances of Log Income, Earnings and Wage Shocks

Authors	Permanent σ^2_ψ	Transitory σ^2_ξ
Individual data		
MaCurdy (1982)[‡]	0.013	0.031
Topel (1991)	0.013	0.017
Topel and Ward (1992)	0.017	0.013
Meghir and Pistaferri (2004)[◇]	0.031	0.032
Nielsen and Vissing-Jorgensen (2006)[¶]	0.005	0.015
Krebs, Krishna, and Maloney (2007)[⋆]	∼ 0.01	∼ 0.1
Jensen and Shore (2008)[◇]	0.054	0.171
Guvenen (2009)	0.015	0.061
Heathcote, Perri, and Violante (2010)[*]	0.01–0.03	0.05–0.1
Hryshko (2012)[◇]	0.038	0.118
Low, Meghir, and Pistaferri (2010)	0.011	–
Sabelhaus and Song (2010)[△]	0.03	0.08
Guvenen, Ozkan, and Song (2012)[∘]	∼ 0.05	∼ 0.125
Karahan and Ozkan (2012)[•]	∼ 0.013	∼ 0.09
Blundell, Graber, and Mogstad (2013)[♣]	∼ 0.015	∼ 0.025
Household data		
Carroll (1992)	0.016	0.027
Carroll and Samwick (1997)	0.022	0.044
Storesletten, Telmer, and Yaron (2004a)	0.017	0.063
Storesletten, Telmer, and Yaron (2004b)	0.008–0.026	0.316
Blundell, Pistaferri, and Preston (2008)[◇]	0.010–0.030	0.029–0.055
Review of Economic Dynamics (2010)[◁]	0.02–0.05	0.02–0.1
Blundell, Low, and Preston (2013)[▷]	∼ 0.005	
DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013)[§]	0.007–0.010	0.15–0.20
Implied by KS-JEDC	0.00	0.038

Notes: [‡]: MaCurdy (1982) did not explicitly separate ψ_t and ξ_t, but we have extracted σ^2_ψ and σ^2_ξ as implications of statistics that his paper reports. First, we calculate $\text{var}(\log \boldsymbol{y}_{t+d} - \log \boldsymbol{y}_t)$ and $\text{var}(\log \boldsymbol{y}_{t+d-1} - \log \boldsymbol{y}_t)$ using his estimate (we set $d = 5$). Then, following Carroll and Samwick (1997) we obtain the values of σ^2_ψ and σ^2_ξ which can match these statistics, assuming that the income process is $\boldsymbol{y}_t = p_t\xi_t$ and $p_t = p_{t-1}\psi_t$ (i.e., we solve $\text{var}(\log \boldsymbol{y}_{t+d} - \log \boldsymbol{y}_t) = d\sigma^2_\psi + 2\sigma^2_\xi$ and $\text{var}(\log \boldsymbol{y}_{t+d-1} - \log \boldsymbol{y}_t) = (d-1)\sigma^2_\psi + 2\sigma^2_\xi$). [◇]: Meghir and Pistaferri (2004), Jensen and Shore (2008), Hryshko (2012), and Blundell, Pistaferri, and Preston (2008) assume that the transitory component is serially correlated (an MA process), and report the variance of a subelement of the transitory component. For example, Meghir and Pistaferri (2004) and Blundell, Pistaferri, and Preston (2008) assume an MA(1) process $\log \xi_t = v_t + \vartheta v_{t-1}$ and obtain estimates (σ^2_v, ϑ)=$(0.0300, -0.2566)$ and $(0.0286–0.0544, 0.1132)$, respectively. σ^2_ξ for these four articles reported in this table are calculated by $(1 + \vartheta^2)\sigma^2_v$ using their estimates. The table does not include Moffitt and Gottschalk (2011) because their income process does not incorporate the MA(1) component; see Appendix B for our estimates of the Moffitt and Gottschalk process. [¶]: Administrative data for Denmark. [⋆]: Data for Mexico, Krebs, Krishna, and Maloney (2007), Table II. [*]: Heathcote, Perri, and Violante (2010), Figure 18. [△]: Sabelhaus and Song (2010), implied by Figure 4. [∘]: Figure 5 of Guvenen, Ozkan, and Song (2012) displays the evolution over time of the standard deviation of the 1-year and 5-year ahead earnings growth, from which we back out the estimates of σ^2_ψ and σ^2_ξ using the above formulas of Carroll and Samwick (1997). [•]: Karahan and Ozkan (2012), Figures 2 and 3, age-invariant model. [♣]: Administrative data for Norway, Blundell, Graber, and Mogstad (2013), Figures 7 and 8. [◁]: Estimates based on data for the US, Canada, the UK, Germany, Italy, Spain, Sweden, Russia and Mexico; see Table 7A–C, pages 11–13 for a summary. [▷]: Data for the UK, Blundell, Low, and Preston (2013), Figure 8. [§]: DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013), Table III, 'Restricted Model'.

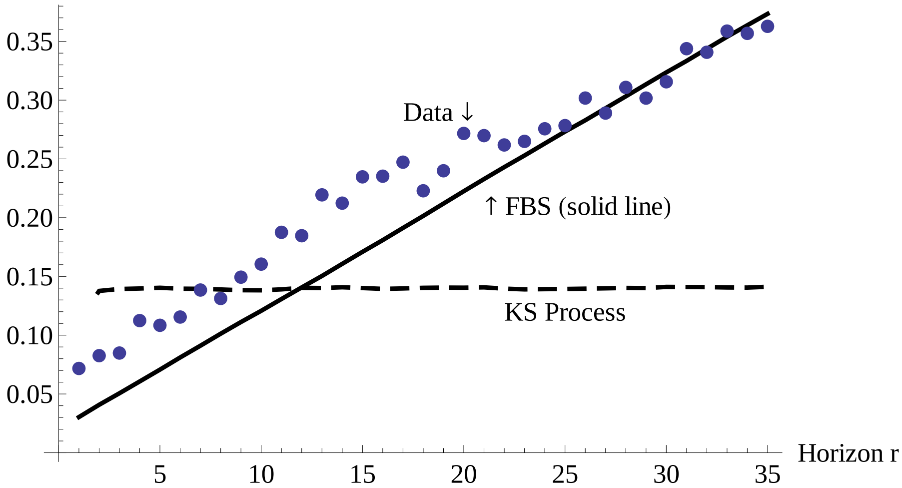

Figure 1 Cross-Sectional Variance of Income Processes and Data, $\text{var}(\log \boldsymbol{y}_{t+r,i} - \log \boldsymbol{y}_{t,i})$

Notes: The data are based on DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013), Figure IV(a) and were normalized so that the variance for $r = 1$, $\text{var}(\log \boldsymbol{y}_{t+1,i} - \log \boldsymbol{y}_{t,i})$ lie in the middle between the values for the KS and the FBS processes.

Figure 1 illustrates further the difficulties of the KS process to match the empirical data on income from DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013).[14] A key feature of the data is that the cross-sectional variance of the income profiles $\text{var}(\log \boldsymbol{y}_{t+r,i} - \log \boldsymbol{y}_{t,i})$ tends to grow linearly with the horizon r, with slope σ_ψ^2. This mirrors closely the characteristics of the FBS process, where $\text{var}(\log \boldsymbol{y}_{t+r,i} - \log \boldsymbol{y}_{t,i}) = 2\sigma_\xi^2 + \sigma_\psi^2 \times r$.[15] In contrast, the statistic for the KS process does not exhibit any trend, also reflecting the fact that the first autocorrelation of the KS income process is only roughly 0.2, contrasting sharply with income processes of Table 1, which are highly persistent.

These discrepancies naturally make one wonder whether the KS-JEDC model's well-known difficulty in matching the degree of wealth inequality is largely explained by its highly unrealistic assumption about the income process.

As a second check of our calibration of the FBS process, we make sure that the parameters of the income process are in line with households' subjective estimates of the permanent income. In particular, since our goal here is to produce a realistic distribution of permanent income across the members of the (simulated) population, we measure the empirical distribution of permanent income in the cross section using data from the Survey of Consumer Finances (SCF), which conveniently includes a question asking

[14] DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013) use a new, rich panel dataset of the Internal Revenue Service. For similar evidence based on the Social Security Administration earnings data see Sabelhaus and Song (2010), Figure 4.

[15] Note that the data were normalized so that the variance for $r = 1$, $\text{var}(\log \boldsymbol{y}_{t+1,i} - \log \boldsymbol{y}_{t,i})$ lie in the middle between the values for the KS and the FBS processes. The key focus of the figure is on the linear trend in the cross-sectional variance rather than on the intercept, for which the empirical estimates vary as given in Table 1, column σ_ξ^2, partly also likely reflecting—beside transitory income shocks—measurement error and initial heterogeneity.

Table 2 Variance of Permanent Income in the Survey of Consumer Finances

Dataset	var(p)	$\mathbb{E}[\psi^2]$	σ^2_ψ
SCF1992	2.5	1.015	0.015
SCF1995	7.5	1.018	0.018
SCF1998	3.1	1.015	0.015
SCF2001	3.6	1.016	0.016
SCF2004	5.2	1.017	0.017
SCF2007	7.3	1.018	0.018
SCF2010	6.4	1.018	0.018
KS-Orig or KS-JEDC	0	1	0

respondents whether their income in the survey year was about 'normal' for them, and if not, asks the level of 'normal' income.[16] This corresponds well with our (and Friedman (1957)'s) definition of permanent income p (and Kennickell (1995) shows that the answers people give to this question can be reasonably interpreted as reflecting their perceptions of their permanent income), so we calculate the variance of $p^i \equiv p^i/\mathbb{M}[p^i]$ among such households.[17]

The results from this exercise are reported in Table 2 (with a final row that makes the point that both the KS model assumes that permanent shocks did not exist). Substituting these estimates for σ^2_p into (7) and (8), we obtain estimates of the variance of ψ. Reassuringly, we can interpret the variances of ψ thus obtained as being easily in the range of the estimated variances of $\log(\psi) = \sigma^2_\psi$ in Table 1.[18] Such a correspondence, across two quite different methods of measurement, suggests there is considerable robustness to the measurement of the size of permanent shocks.

3.2 Time Preference Factor

This section calibrates the time preference factor. As a preliminary theoretical consideration, note that the following 'Death-Modified' extension of the Carroll (2011) 'Growth Impatience Condition' (which generalizes Deaton (1991) and Bewley (1977)) ensures that models of this kind have a well-defined solution for infinite-horizon consumers (see Appendix C for details):

$$\frac{(\mathsf{R}\beta)^{1/\rho}\,\mathbb{E}[\psi^{-1}]\not{D}}{\Gamma} \;<\; 1. \tag{16}$$

Carroll (2011) dubs this inequality the 'Growth Impatience Condition' because it guarantees that consumers are sufficiently impatient to prevent the indefinite increase in the *ratio* of net worth to (stochastically growing) permanent income (see also Szeidl

[16]SCF1992 only asked whether the income level was about 'normal' or not.

[17]We restrict the sample to households between the ages of 25 and 60, because the interpretation of the question becomes problematic for retired households.

[18]So long as the variance of the permanent shocks is small, these two measures should be approximately the same.

Table 3 Parameter Values and Steady State

Description	Parameter	Value	Source
Representative agent model			
Time discount factor	β	0.99	JEDC (2010)
Coefficient of relative risk aversion	ρ	1	JEDC (2010)
Capital share	α	0.36	JEDC (2010)
Depreciation rate	δ	0.025	JEDC (2010)
Time worked per employee	ℓ	1/0.9	JEDC (2010)
Steady state			
Capital–output ratio	$\boldsymbol{K}/\boldsymbol{Y}$	10.26	JEDC (2010)
Effective interest rate	$\mathsf{r}-\delta$	0.01	JEDC (2010)
Wage rate	W	2.37	JEDC (2010)
Heterogenous agents models			
Unemployment insurance payment	μ	0.15	JEDC (2010)
Unemployment rate	u	0.07	Mean in JEDC (2010)
Probability of death	D	0.00625	Yields 40-year working life
Variance of log $\theta_{t,i}$	σ_θ^2	0.010×4	Carroll (1992)
Variance of log $\psi_{t,i}$	σ_ψ^2	0.010/4	Carroll (1992)
			DeBacker et al. (2013)
KS aggregate shocks			
Shock to productivity	$\triangle^Z$	0.01	Krusell and Smith (1998)
Unemployment (good state)	u^g	0.04	Krusell and Smith (1998)
Unemployment (bad state)	u^b	0.10	Krusell and Smith (1998)
Aggregate transition probability		0.125	Krusell and Smith (1998)

Notes: The models are calibrated at the quarterly frequency. The steady state values are calculated on a quarterly basis.

(2012)). This condition is an amalgam of the pure time preference factor, expected growth, the relative risk aversion coefficient, probability of surviving and the real interest factor. Thus, a consumer can be 'impatient' in the required sense even if $\beta = 1$, so long as expected income growth is positive.[19]

We search for the time preference factor $\grave{\beta}$ such that if all households had an identical $\beta = \grave{\beta}$ the steady-state value of the capital-to-output ratio ($\boldsymbol{K}/\boldsymbol{Y}$) would match the value that characterized the steady-state of the perfect foresight model.[20] $\grave{\beta}$ turns out to be 0.9888 (recall that this is at a quarterly, not an annual, rate).

3.3 Other Parameters

Except where otherwise noted, our remaining parametric assumptions match those of the papers in the special JEDC volume (cited above).[21] Henceforth, we refer to the version of the model solved by the papers in the special JEDC volume as the 'KS-JEDC' model. The parameters are reproduced for convenience in the top panel of Table 3.[22]

When aggregate shocks are shut down ($Z_t = 1$ and $\boldsymbol{L}_t = \boldsymbol{L}$), the model has a steady-state solution with a constant ratio of capital to output and constant interest and wage rates, which we write without time subscript as r and W and which are reflected in Table 3.[23]

Three parameters characterize our modifications to the KS-JEDC model: D, σ_θ^2, and σ_ψ^2. The probability of dying $\mathsf{D} = 0.00625$ implies the average length of working life is $1/0.00625 = 160$ quarters $= 40$ years (dating from entry into the labor force at, say, age 25). The variances of log transitory income shocks $\sigma_\theta^2 = 0.010$ and log permanent income shocks $\sigma_\psi^2 = 0.010$ are the values advocated in Carroll (1992) (based on the Panel Study of Income Dynamics (PSID) data).[24] The latter number also closely mirrors the new estimates of DeBacker, Heim, Panousi, Ramnath, and Vidangos (2013) on the high-quality tax data from the IRS.

4 Matching the Wealth Distribution

We now ask whether our model with realistically calibrated income and finite lifetimes can reproduce the degree of wealth inequality evident in the micro data.[25] An improvement in the model's ability to match the data (over the KS model) is to be expected, since in buffer stock models agents strive to achieve a target *ratio* of wealth to permanent income. By assuming no dispersion in the level of permanent income across households, KS's income process disables a potentially vital explanation for variation in the level of target wealth (and, therefore, on average, actual wealth) across households.

Table 4 shows that the models with the FBS income process do indeed yield a substantial improvement over the distribution of net worth implied by our solution of

[19]This near-equivalence explains why we do not bother to include a growth term in the process for noncapital income in (8)–(10) despite the presence of such a term in (6); inclusion of the income growth term should mostly just result in an offsetting effect on our estimated time preference rate, and would complicate our simulations unnecessarily.

[20]Output is the sum of noncapital and capital income.

[21]Examples of such authors include Young (2010) and Algan, Allais, and Den Haan (2008).

[22]The only effective difference between the 'KS-JEDC' model and the original Krusell and Smith (1998) model is the introduction (for realism) of unemployment insurance in the KS-JEDC version, which does not matter much for any substantive results. To be very precise, another difference is the introduction of ℓ (time worked per employee) in the KS-JEDC model, but this does not have a real impact.

[23]In the steady state, $\boldsymbol{K}_t/(\ell \boldsymbol{L}_t) = \bar{k} = (\alpha\beta/(1-\beta(1-\delta)))^{1/(1-\alpha)} = 38.0$, r (gross interest rate) $= \alpha\bar{k}^{\alpha-1}$, and $\mathsf{W} = (1-\alpha)\bar{k}^{\alpha}$.

[24]This paper assumes that each period corresponds to a *quarter*, while $\sigma_\theta^2 = 0.010$ from Carroll (1992) is the value on an annual basis. Therefore, following Carroll, Slacalek, and Tokuoka (2008), 0.010 needs to be multiplied by 4 since the variance of log transitory income shocks of *quarterly* data should be four times as large as that of annual data. (Note further that Carroll (1992)'s calibration of $\sigma_\theta^2 = 0.010$ was considerably lower than his raw empirical estimate of 0.027, on the grounds that a substantial portion of the changes in measured income is likely to come from measurement error). Since σ_ψ^2 (0.010) is also an annual variance, it needs to be *divided* by 4, following Carroll, Slacalek, and Tokuoka (2008).

[25]Throughout this paper, we will examine the distribution of net worth (not financial or gross assets).

Table 4 Proportion of Net Worth by Percentile in Models and the Data (in Percent)

	Income Process			
Percentile of	KS-JEDC	Friedman/ Buffer Stock‡		
Net Worth	Our Solution	No Aggr Unc	KS Aggr Unc	Data*
Top 1%	3.0	10.0	7.9	33.9
Top 10%	22.9	38.0	34.2	69.7
Top 20%	39.7	55.1	51.2	82.9
Top 40%	65.4	76.9	73.8	94.7
Top 60%	83.5	90.1	88.2	99.0
Top 80%	95.1	97.5	96.8	100.2

Notes: $\boldsymbol{K}_t/\boldsymbol{Y}_t = 10.3$. ‡ : $\grave{\beta} = 0.9888$. * : The data is the SCF 2004.

the KS-JEDC model solved without an aggregate shock (or the results of the original Krusell and Smith (1998) model);[26] compare the models in columns 2 and 3, and the KS model in column 1 to the data in the last column. For example, in our model with the FBS income and no aggregate uncertainty, the fraction of total net worth held by the top 1 percent is about 10 percent, while the corresponding statistic is only 3 percent in our solution of the KS-JEDC model.

The KS-JEDC model's failure to match the wealth distribution is not confined to the top. In fact, perhaps a bigger problem is that the model generates a distribution of wealth in which most households' wealth levels are not very far from the wealth target of a representative agent in the perfect foresight version of the model. For example, in steady state about 50 percent of all households in the KS-JEDC model have net worth between 0.5 times mean net worth and 1.5 times mean net worth; in the SCF data from 1992–2004, the corresponding fraction ranges from only 20 to 25 percent.

But while our model fits the data better than the original KS model, it still falls short of matching the empirical degree of wealth inequality. The proportion of net worth held by households in the top 1 percent of the distribution is three times smaller in the model than in the data (compare the second and last columns in the table). This failure reflects the fact that, empirically, the distribution of wealth is considerably more unequal than the distribution of permanent income.

In this paper, we do not attempt to further improve how the model with the FBS income matches the wealth distribution, but Carroll, Slacalek, and Tokuoka (2013) show that doing so is straightforward by adding modest heterogeneity in impatience. Specifically, Carroll, Slacalek, and Tokuoka (2013) estimate that a model with discount factors distributed uniformly between roughly 0.98 and 0.99 fits the empirical wealth distribution. (And the original Krusell and Smith (1998) paper showed that the 'stochastic-

[26] Our solution of the KS-JEDC model is very similar to the results of the original KS model in terms of wealth distribution; what small differences do exist reflect the minor difference in the assumption about unemployment insurance (discussed earlier) as well as the fact that the original KS model was solved with aggregate shocks turned on.

β' in which the discount factor follows a three-state Markov process, does a much better job of matching the wealth distribution; see Carroll, Slacalek, and Tokuoka (2013) for further discussion.)

5 Model with KS Aggregate Shocks

This section examines a model with an FBS household income process that also incorporates aggregate shocks of the kind KS included, and investigate the model's performance in matching the wealth distribution and in replicating aggregate statistics.

Krusell and Smith (1998) assumed that the level of aggregate productivity alternates between $Z_t = 1+\triangle^Z$ if the aggregate state is good and $Z_t = 1-\triangle^Z$ if it is bad; similarly, $\boldsymbol{L}_t = 1 - u_t$, where $u_t = u^g$ if the state is good and $u_t = u^b$ if bad. (For reference, we reproduce their assumed parameter values in the bottom panel of Table 3 above.)

The decision problem for an individual household in period t can be written using normalized variables and the employment status ι_t:

$$\begin{aligned}
\mathrm{v}(m_t, \iota_t; \boldsymbol{K}_t, Z_t) &= \max_{c_t} \ \mathrm{u}(c_t) + \beta \not{D} \mathbb{E}_t \left[(\Gamma_{t+1}\psi_{t+1})^{1-\rho} \mathrm{v}(m_{t+1}, \iota_t; \boldsymbol{K}_{t+1}, Z_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t, \\
a_t &\geq 0, \\
k_{t+1} &= a_t / (\not{D}\Gamma_{t+1}\psi_{t+1}), \\
m_{t+1} &= ((1-\delta) + r_{t+1})k_{t+1} + y_{t+1}, \\
r_{t+1} &= \alpha Z_{t+1} (\boldsymbol{K}_{t+1}/\ell \boldsymbol{L}_{t+1})^{\alpha - 1},
\end{aligned} \tag{17}$$

where

- the non-bold *individual* variables (lower-case variables except for ι_t and ψ_t) are the bold (level) variables divided by $\mathrm{A}_t \boldsymbol{p}_t$ (e.g., $a_t = \boldsymbol{a}_t / \mathrm{A}_t \boldsymbol{p}_t$),
- $\Gamma_{t+1} = \mathrm{A}_{t+1}/\mathrm{A}_t$,
- $\boldsymbol{L}_t = 1 - u_t$, and
- the income process is the same as in (8)–(12) but the employment transition process follows KS-JEDC.

There are more state variables in this version of the model than in the model with no aggregate shock: The aggregate variables Z_t and $\boldsymbol{K}_t$, and the household's employment status ι_t whose transition process depends on the aggregate state. Solving the full version of the model above with both aggregate and idiosyncratic shocks is not straightforward; the basic idea for the solution method is the key insight of Krusell and Smith (1998). See Appendix D for details about our solution method.

A comparison of columns 2 and 3 of Table 4 show that the model with KS aggregate shocks and FBS idiosyncratic shocks does roughly the same job matching the wealth

Table 5 Aggregate Statistics

	Income Process			
	KS Agg Unc Only	KS-JEDC Our Solution	Friedman/Buffer Stock KS Agg Unc	Data
$\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$	0.24	0.23	0.13	0.51
$\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{Y}_t)$	0.84	0.86	0.92	0.50
$\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{Y}_{t-1})$	0.15	0.15	0.11	0.31
$\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{Y}_{t-2})$	0.11	0.13	0.09	0.17
$\varrho(\Delta \log \boldsymbol{C}_t, \mathsf{r}_t)$	0.86	0.85	0.77	0.27
$\varrho(\Delta \log \boldsymbol{C}_t, \mathsf{r}_{t-1})$	0.28	0.26	0.12	0.20
$\varrho(\Delta_4 \log \boldsymbol{C}_t, \Delta_4 \log \boldsymbol{Y}_t)$	0.67	0.70	0.81	0.76
$\varrho(\Delta_8 \log \boldsymbol{C}_t, \Delta_8 \log \boldsymbol{Y}_t)$	0.61	0.63	0.75	0.87

Notes: Δ_4 and Δ_8 are one-year and two-year growth rates, respectively. The statistics for the US data in column 4 were calculated for the range 1960Q1–2011Q4.

distribution as the FBS model without aggregate shocks. The fact that the specification of aggregate shock affects little the performance of the model in this respect is not surprising because it is well-known that aggregate shocks are much smaller than idiosyncratic shocks. (Related to this, the literature tends to agree that in this class of models the welfare cost of business cycles is low, see the large literature starting with Lucas (1985).)[27]

Table 5 reports statistics on aggregate dynamics for the following models with aggregate shocks: the representative agent model with KS aggregate shocks and no idiosyncratic uncertainty (column 1); our solution of the KS-JEDC model (column 2); and the model with the FBS idiosyncratic shocks and the KS aggregate shocks (column 3), and compares these statistics to the US aggregate data (column 4). The results are generally similar across all models implying positive autocorrelation of consumption growth, and high contemporaneous correlation of consumption growth with income growth and interest rates. The serial correlation of consumption growth in our solution of the KS-JEDC model, 0.23, is similar to the value 0.28 reported for the KS-JEDC model by Maliar, Maliar, and Valli (2008).[28,29]

For the autocorrelation of the consumption growth the KS-JEDC model exhibits a relatively high value, which is closer to the US data (where for non-durables and services consumption the statistic is about 0.5) than the value implied by consumption models stemming from Hall (1978). At first blush, it seems surprising that the KS-JEDC model, which includes neither habits nor sticky expectations, substantially violates the random

[27]Adding aggregate shocks slightly worsens the model's fit of the top tail of the wealth distribution because all households increase somewhat their saving for precautionary reasons.

[28]The difference between the results in Maliar, Maliar, and Valli (2008) and ours reflects approximation error in solving the consumption function.

[29]Although not reported here, our solution of the KS-JEDC model closely matches theirs in other aggregate statistics as well (e.g., variance of aggregate consumption (level), correlation between income and consumption levels).

walk proposition, a puzzle that has not been noticed in the previous literature on the Krusell and Smith (1998) model (so far as we know). With additional simulations (see Appendix E for details), we have found that appearance of sticky consumption growth in the KS-JEDC model actually results from the high degree of serial correlation in interest rates implied by the assumption about the process for aggregate productivity shocks. The omission of the component of consumption growth that is predictable from the interest rate accounts for the apparent violation of the random walk proposition.

6 Conclusion

We see the virtues of our approach as three. First, we have resolved the longstanding question of how much difference (quantitatively) it would make to incorporate a quantitatively realistic (but still simple) microeconomic income process in a Krusell and Smith (1998)-type model. Second, we have shown that while the incorporation makes little difference to macroeconomic statistics like covariances or serial correlation, the model with permanent shocks goes some way toward making the baseline model (without time preference heterogeneity) more consistent with the large degree of wealth heterogeneity in the population. Finally, our model is substantially simpler and easier to solve and simulate than the original Krusell and Smith (1998) model, which should make it easier to adopt for future research.

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Appendix

A Cross-Sectional Variance of Permanent Income

The evolution of the square of p is given by

$$\begin{aligned} p_{t+1,i} &= p_{t,i}\psi_{t+1,i}(1-\mathsf{d}_{t+1,i}) + \mathsf{d}_{t+1,i}, \\ p^2_{t+1,i} &= \big(p_{t,i}\psi_{t+1,i}(1-\mathsf{d}_{t+1,i})\big)^2 + 2p_{t,i}\psi_{t+1,i}\underbrace{\mathsf{d}_{t+1,i}(1-\mathsf{d}_{t+1,i})}_{=0} + \mathsf{d}^2_{t+1,i}, \end{aligned}$$

where $\mathsf{d}_{t+1,i} = 1$ if household i dies.

Because $\mathbb{E}_t[(1-\mathsf{d}_{t+1,i})^2] = 1-\mathsf{D}$ and $\mathbb{E}_t[\mathsf{d}^2_{t+1,i}] = \mathsf{D}$, we have

$$\begin{aligned} \mathbb{E}_t[p^2_{t+1,i}] &= \mathbb{E}_t[(p_{t,i}\psi_{t+1,i}(1-\mathsf{d}_{t+1,i}))^2] + \mathsf{D}, \\ &= p^2_{t,i}\cancel{\mathsf{D}}\mathbb{E}[\psi^2] + \mathsf{D} \end{aligned}$$

and

$$\mathbb{M}\left[p^2_{t+1}\right] = \mathbb{M}[p^2_t]\cancel{\mathsf{D}}\mathbb{E}[\psi^2] + \mathsf{D}.$$

Finally, the steady state expected level of $\mathbb{M}[p^2] \equiv \lim_{t\to\infty}\mathbb{M}[p^2_t]$ can be found from the equation $\mathbb{M}[p^2] = \mathsf{D} + \cancel{\mathsf{D}}\mathbb{E}[\psi^2]\mathbb{M}[p^2]$:

$$\mathbb{M}[p^2] = \frac{\mathsf{D}}{1-\cancel{\mathsf{D}}\mathbb{E}[\psi^2]}.$$

B Estimating the Moffitt and Gottschalk (2011) Income Process on Simulated FBS Data

This appendix estimates the *annual* income process specified by *à la* Moffitt and Gottschalk (2011) using simulation results of a *quarterly* quarterly data generated by our FBS income process (with parameter values from Table 3). Moffitt and Gottschalk (2011) assume log permanent income $\log(p_t)$ follows a random walk and log transitory income $\log(\xi_t)$ follows an ARMA process at the annual frequency:

$$\begin{aligned} \boldsymbol{y}_t &= p_t\xi_t, \\ \log(p_t) &= \log(p_{t-1}) + \log(\psi_t), \\ \log(\xi_t) &= a_1\log(\xi_{t-1}) + v_t + m_1 v_{t-1}. \end{aligned}$$

Like Moffitt and Gottschalk (2011), we match the covariance matrix of the annual income draws, and obtain estimates with the same signs, and similar magnitudes, to those they obtain using the PSID data; see Table 6, confirming that our calibration is qualitatively consistent with Moffitt and Gottschalk's.

An interesting result is that even though our true quarterly transitory shock process is just white noise, if we estimate the process on an annual basis we obtain positive AR (a_1) and

Table 6 Estimates of Moffitt and Gottschalk Annual Income Process on Simulated FBS Data

	σ^2_ψ	σ^2_v	a_1	m_1
Our estimates	0.009	0.025	0.578	−0.613
Moffitt and Gottschalk (2011)	0.00159	0.169	0.622	−0.344

negative MA (m_1) coefficients. This suggests that the positive a_1 and negative m_1 reported in Moffitt and Gottschalk (2011) may be (at least) partly due to time aggregation.

C Death-Modified 'Growth Impatience Condition'

In the model normalized by permanent income, the return factor for the consumers who live is

$$\begin{aligned}
\mathbb{E}_t[\mathcal{R}_{t+1}] &= \mathbb{E}_t[\psi^{-1}_{t+1}\mathsf{R}_{t+1}/(\cancel{D}\Gamma)], \\
&= \mathbb{E}[\psi^{-1}]\mathsf{R}/(\cancel{D}\Gamma), \\
&\equiv \cancel{D}^{-1}\underbrace{\mathbb{E}[\psi^{-1}]\mathsf{R}/\Gamma}_{\mathbf{R}},
\end{aligned}$$

where $\cancel{D}$ is the probability of surviving, Γ is the underlying growth rate of permanent income, and we will be looking for the steady state, where $\mathsf{R}_{t+1} = \mathsf{R}$ is constant.

For the consumers who live,

$$\begin{aligned}
\mathbb{E}_t[m_{t+1}|\text{Live}] &= \mathbb{E}_t[(m_t - c_t)\,\mathcal{R}_{t+1} + \xi_{t+1}], \\
&= m_t\cancel{D}^{-1}\mathbf{R} - c_t\cancel{D}^{-1}\mathbf{R} + 1,
\end{aligned}$$

while for those who die,

$$\mathbb{E}_t[m_{t+1}|\text{Die}] = 1,$$

so for a population of households alive at date t the overall expectation weights those who live by $\cancel{D}$ and those who die by $(1-\cancel{D})$:

$$\begin{aligned}
\mathbb{E}_t[m_{t+1}] &= \cancel{D}(m_t \overbrace{\cancel{D}^{-1}\mathbf{R}}^{=\mathbb{E}_t[\mathcal{R}_{t+1}]} - c_t\cancel{D}^{-1}\mathbf{R} + 1) + (1-\cancel{D}), \\
&= (m_t - c_t)\mathbf{R} + 1,
\end{aligned}$$

which is the *same* locus as in the model without death. From this we can derive

$$\mathbb{E}_t[\Delta m_{t+1}] = m_t(\mathbf{R} - 1) - c_t\mathbf{R} + 1,$$

so that the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus is

$$c_t = \frac{1}{\mathbf{R}} + \frac{\mathbf{R} - 1}{\mathbf{R}} m_t$$

and has the slope $\frac{\mathbf{R}-1}{\mathbf{R}}$.

Now, as $m_t \to \infty$, the slope of the consumption function converges to that of the perfect foresight case with the effective discount factor $\not{D}\beta$ and the effective interest rate $\not{D}^{-1}\mathsf{R}$ (Carroll (2011)):

$$1 - \frac{\left((\not{D}\beta)(\not{D}^{-1}\mathsf{R})\right)^{1/\rho}}{\not{D}^{-1}\mathsf{R}} = 1 - \frac{(\beta\mathsf{R})^{1/\rho}}{\not{D}^{-1}\mathsf{R}}.$$

Stationary distribution of wealth requires that the consumption function exceeds eventually the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus. 'The Death-Modified Growth Impatience Condition,' which ensures this happens, is that as $m_t \to \infty$, the slope of the consumption function exceeds that of the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus (Carroll (2011)):

$$\begin{aligned} 1 - \frac{(\beta\mathsf{R})^{1/\rho}}{\not{D}^{-1}\mathsf{R}} &> \frac{\mathbf{R}-1}{\mathbf{R}} \\ \Leftrightarrow \frac{(\beta\mathsf{R})^{1/\rho}\,\mathbb{E}[\psi^{-1}]\not{D}}{\Gamma} &< 1. \end{aligned}$$

D Solution Algorithm

In solving the problem in section 5 we closely follow the stochastic simulation method of Krusell and Smith (1998). Krusell and Smith find that per capita capital today ($\boldsymbol{K}_t$) is sufficient to predict per capita capital tomorrow ($\boldsymbol{K}_{t+1}$). Our procedure is as follows:

1. Solve for the optimal individual decision rules given some 'beliefs' π that determine the (expected) law of motion of per capita capital. The law of motion is takes the log-linear form given by $\pi = (\pi_0, \pi_1, \pi_0', \pi_1')$:

$$\log \boldsymbol{K}_{t+1} = \pi_0 + \pi_1 \log \boldsymbol{K}_t$$

   if the aggregate state in period t is good ($Z_t = 1 + \triangle^Z$), and

$$\log \boldsymbol{K}_{t+1} = \pi_0' + \pi_1' \log \boldsymbol{K}_t$$

   if the aggregate state is bad ($Z_t = 1 - \triangle^Z$).

2. Simulate the economy populated by 8,000 households[30] (which experiments determined is enough to suppress idiosyncratic noise) for 1,100 periods (following Maliar, Maliar, and Valli (2010)). When starting a simulation, $p_{t,i} = 1$ for all i, the distribution of $m_{t,i}$ is generated assuming $k_{t,i}$ is equal to its steady state level (38.0) for all i, and $Z_t = 1 + \triangle^Z$ (the aggregate state is good). (The steady state level of $k_{t,i}$ is $\bar{k} = (\alpha\beta/(1-\beta(1-\delta)))^{1/(1-\alpha)}$. With $k_{t,i} = 38.0$ for all i, $\boldsymbol{k}_{t,i} = \boldsymbol{K}_t = 41.2$.) The newborn households start life with $p_{t,i} = 1$ and $k_{t,i} = 0$.

3. Estimate $\tilde{\pi}$, which determines the law of motion of per capita capital, using the last 1,000 periods of data generated by the simulation (we discard the first 100 periods).

4. Compute an improved vector for the next iteration by $\hat{\pi} = (1-\eta)\tilde{\pi} + \eta\pi$ with $\eta = 3/4$.

[30] In the model with the FBS income process.

We repeat this process until $\hat{\pi} = \pi$ with a given degree of precision.[31]

From the second iteration and thereafter, we use the terminal distribution of wealth (and permanent component of income (p)) in the previous iteration as the initial one.

While we can eventually obtain some solution whatever the initial π is, we use π obtained using the representative agent model as the starting point. This can significantly reduce the time needed to obtain the solution.

Parameter values to solve the model are from Table 3. The time preference factors are imposed to be those estimated in section 3.2.

D.1 Tricks to Reduce Simulation Errors

In obtaining the aggregate law, we introduce the following tricks to reduce simulation errors (or to speed up the solution given a degree of estimate precision):

- **Death:** When death is concentrated among households at the very top of the wealth distribution, per capita capital would be at a lower than normal level. To alleviate simulation errors from this source, each period we: i) sort households by wealth level, ii) construct groups, the size of which is the inverse of the death probability (under our parameter choice, the size of each group is 160 and the first group contains households from the wealthiest to the 160th), and iii) pick one household that dies within each group.

- **Permanent income shocks:** In our methodology, permanent shocks to income are approximated by n discrete points. Similarly to the death element, after sorting we set up groups each of size n. We randomize shocks within each group subject to the constraint that each shock point is experienced by one of the group members every period, making the group mean of the shocks equal to the theoretical mean.[32]

D.2 Estimated Laws of Motion

The estimated laws of motions are given in Table 7. The fit measured with R^2 in all specifications exceeds 0.9999.[33]

E Understanding Sticky Consumption Growth in the KS-JEDC Model

Although $\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$ reported in section 5 may not be high enough relative to that observed in the US data, it is still not clear why simulations produce such a high value. Previous studies on KS type models have not investigated this issue. Using the KS-JEDC model, we performed an experiment to understand the phenomenon better. In this experiment we assume that the aggregate state switches from good to bad (or from bad to good) every eight quarters.[34]

[31] In our analysis below, the process is iterated until the difference between each estimate (π_0, π_1, π_0', or π_1') and its previous value is smaller than 1 percent.

[32] This idea is motivated by Braun, Li, and Stachurski (2009), who proposed the estimation of densities with smaller simulation errors by calculating conditional densities given simulated data.

[33] Note that, as pointed out by Den Haan (2010), R^2 only measures in-sample fit and should be interpreted with caution.

[34] Because one state switches to another with a probability of 0.125, the average length of each state is eight quarters in typical simulation.

Table 7 Estimated Laws of Motion

$\log \boldsymbol{K}_{t+1} = \pi_0 + \pi_1 \log \boldsymbol{K}_t + \epsilon_{t+1}$

Model	FBS Income Process		KS-JEDC	
State	Good	Bad	Good	Bad
π_0	0.140	0.126	0.138	0.122
π_1	0.963	0.965	0.963	0.966

Notes: The coefficients for the KS-JEDC model are very close to those estimated in Maliar, Maliar, and Valli (2010).

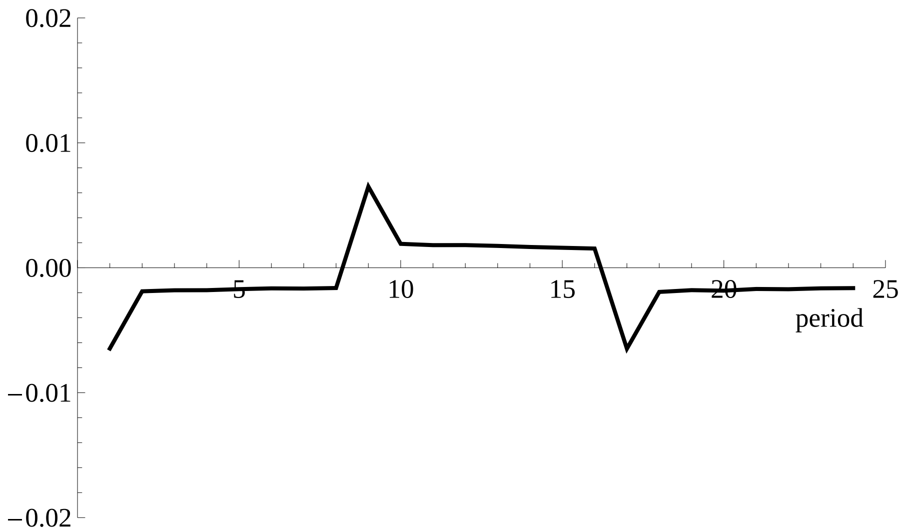

Figure 2 Dynamics of $\Delta \log \boldsymbol{C}_t$ in KS-JEDC Model

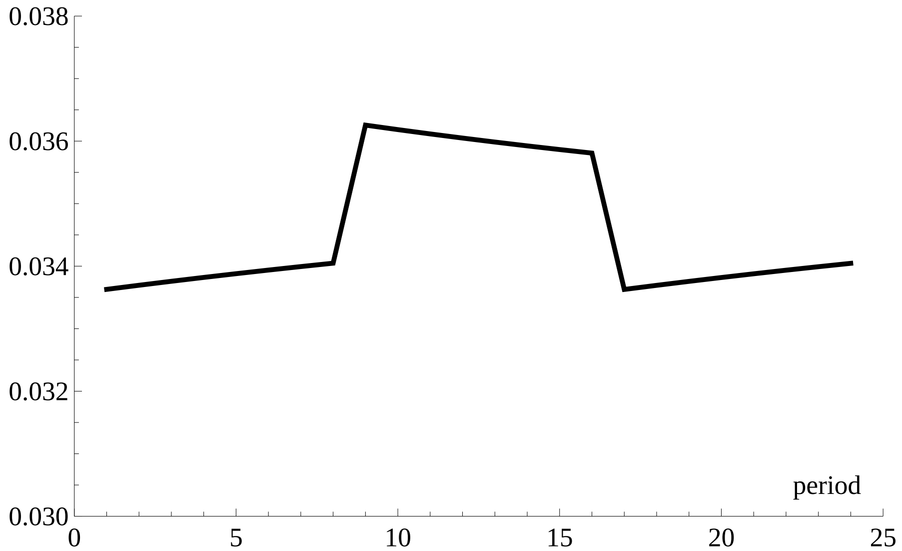

Figure 3 Dynamics of r_t in KS-JEDC Model

Figure 2 plots $\Delta \log \boldsymbol{C}_t$ 24 quarters of simulated observations (the state is bad for the first eight quarters, good for the next eight quarters, and bad for the final eight quarters). The figure shows that $\Delta \log \boldsymbol{C}_t$ is very persistent (it is negative in the bad state and positive in the good state), resulting in a relatively high $\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$.

It is easy to understand that $\Delta \log \boldsymbol{C}_t$ is higher when the state is good (and vice versa) given the following facts:

- A first order approximation of the Euler equation yields:

$$\Delta \log \boldsymbol{C}_t \approx b_0 + b_1 r_t, \tag{18}$$

  where $b_0 \equiv -\rho^{-1}(1 - \beta + \delta)$, $b_1 \equiv \rho^{-1}$, ρ is the coefficient of relative risk aversion, r_t is the interest rate, β is the time preference factor, and δ is the depreciation rate. Indeed, when we conduct an IV regression of equation (18) using r_{t-1} as the instrument,[35] which effectively means estimating $\Delta \log \boldsymbol{C}_t = b_0 + b_1 \mathbb{E}_{t-1}[r_t] + \varepsilon_t$, the estimate of $b_1 \equiv \rho^{-1}$ is 0.95 (with a standard deviation of 0.08) and relatively close to the actual value of ρ^{-1} $(= 1)$. This suggests that using the predictable component of interest rates $(\mathbb{E}_{t-1}[r_t])$, we can obtain a reasonable estimate of intertemporal elasticity of substitution.

- When the state is good, $r_t = \alpha Z_t(\boldsymbol{K}_t/\ell \boldsymbol{L}_t)^{\alpha-1}$ (from (17)) is higher because Z_t (aggregate productivity) is higher, as can be seen in Figure 3, which plots the dynamics of r_t for the 24 quarters.

While in typical simulation one state does not generally last for exactly eight quarters, we observe sticky aggregate consumption growth (and a relatively high $\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$) because the same mechanisms are at work as in the experiment above.

In sum, a relatively high $\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$ in the KS-JEDC model can be interpreted as a consequence of the persistent behavior of the interest rate r_t. Indeed, denoting $\varepsilon_t = \Delta \log \boldsymbol{C}_t - b_0 - b_1 \mathbb{E}_{t-1}[r_t]$ the residual after controlling for the predictable component of consumption growth related to interest rates, we find that $\varrho(\varepsilon_t, \varepsilon_{t-1}) = 0.01$ is much lower than $\varrho(\Delta \log \boldsymbol{C}_t, \Delta \log \boldsymbol{C}_{t-1})$.[36]

[35] The data that produced Table 5 are used.

[36] Estimating an AR(1) process on ε_t produces a small and statistically insignificant coefficient on lagged ε_{t-1}.