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January 2002

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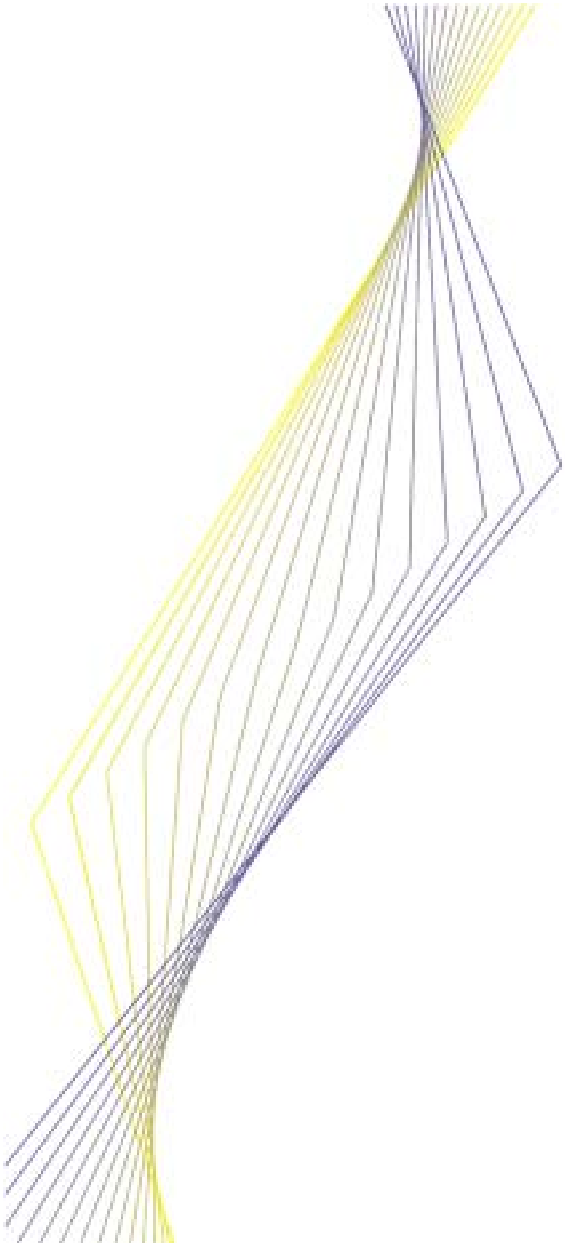

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The views expressed in this paper are those of the authors and do not necessarily reflect those of the European Central Bank.

* *This work was in part carried out when the first author visited the European Central Bank's Directorate General Research as part of their Research Visitor Programme and the second author the Research Department of the Bank of Finland. Financial support from Academy of Finland, Yrjö Jahnsson Foundation and Nokia Group is also gratefully acknowledged.*

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ISSN 1561-0810

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Abstract

An economy exhibits structural heterogeneity when the forecasts of different agents have different effects on the determination of aggregate variables. We study how different forms of heterogeneity in structure, forecasts and adaptive learning rules affect the conditions for convergence of adaptive learning towards rational expectations equilibrium. Results are applied to the market model with supply lags, a New Keynesian model of interest rate setting and the monetary inflation model with heterogenous agents.

Key words: Adaptive learning, expectations formation, stability of equilibrium, market model, monetary policy, Cagan model

JEL classification: D83, C62, E30

Non-Technical Summary

There has been a large amount of research into the implications of adaptive learning behavior in expectations formation for economic dynamics. Paralleling general macroeconomics, most of the research that uses adaptive learning has been carried out in models with representative agents, i.e. in economies with *structural homogeneity*. In studies of adaptive learning the assumption of a representative agent is usually interpreted to mean that expectations and learning rules are also identical. These kinds of assumptions are made mostly for analytical convenience rather than for their realism. In this paper we reconsider stability of rational expectations equilibrium (REE) under adaptive learning when the economy exhibits *structural heterogeneity,* i.e. the assumption of structural homogeneity is relaxed.

In economies with structural heterogeneity the basic characteristics differ across consumers (and firms), so that they respond to expectations in different ways. If this is the case, it is natural to assume that expectations of different agents can also differ. We will make the further distinction that heterogeneity in expectations can be due to different initial beliefs or the use of different learning algorithms by the agents. Clearly, structurally homogenous economies can exhibit heterogenous expectations (and this possibility is permitted in some studies, see the references below), but different agents respond to expectations in the same way in economies with structural homogeneity.

While different approaches to adaptive learning exist, probably the largest concentration of research has used what is called the statistical or econometric learning. In this approach the economic agents are assumed to use forecast functions that depend on some parameters and, at any moment of time, the economic decisions are made on the basis of expectations/forecasts obtained from these functions. The values of the parameters in these functions and expectations of the agents are adjusted over time as new data becomes available. Parameter updating is assumed to be done using standard econometric methods such as recursive least squares (RLS) estimation. A key issue of interest is whether this kind of adaptive learning behavior converges to a rational expectations equilibrium (REE) over time. If this is the case, then eventually the forecast functions of the agents are those associated with the REE.

In this paper our goal is to consider the stability of REE when both structural and expectational heterogeneity is present. The basic framework will be a multivariate linear model with two classes of agents. While the assumption of linearity is directly postulated for some models in the literature, we can also justify this assumption by observing that most applied studies are in any case based on linearization. The restriction to two classes of agents in the main analysis is done only for simplicity of formulation, and we also state the stability conditions for economies with a finite number of different classes of agents. As already noted, heterogenous expectations can arise because of different initial beliefs or because the learning rules of the agents differ and we will analyze both possibilities. The case of different learning rules is further subdivided into (a) situations where the learning rules of the different agents are the same, except for their degree of responsiveness to forecast errors and (b) when the learning rules take different forms. In case (b) we assume that some agents use the RLS algorithm while others use the stochastic gradient algorithm. We also take up the case in which one class of agents has continually RE while others are learning as this case occasionally appears in the literature.

The economy may be purely forward looking or it may also include lags of endogenous variables. Though we will work out the details in the forward looking framework, the main analytical steps for economies with lags will also be outlined. General convergence conditions are derived in the different cases of heterogeneity. These conditions are used in three economic applications: (i) Muth's market model, (ii) a model of monetary policy and (iii) Cagan's inflation model. It is shown that structural heterogeneity does not affect the stability conditions in models (i) and (iii). However, for model (ii) structural heterogeneity can matter and the convergence conditions of the earlier literature are only necessary, but not in general sufficient for convergence of heterogenous learning to the REE. (A more detailed analysis of model (ii) is done in a companion paper.)

1 Introduction

There has been a large amount of research into the implications of adaptive learning behavior in expectations formation for economic dynamics.[1] Paralleling general macroeconomics, most of the research that uses adaptive learning has been carried out in models with representative agents, i.e. in economies with *structural homogeneity*. In studies of adaptive learning the assumption of a representative agent is usually interpreted to mean that expectations and learning rules are also identical. These kinds of assumptions are made mostly for analytical convenience rather than for their realism. In this paper we reconsider stability of REE under adaptive learning when the economy exhibits *structural heterogeneity,* i.e. the assumption of structural homogeneity is relaxed.[2]

In economies with structural heterogeneity the basic characteristics differ across consumers (and firms), so that they respond to expectations in different ways. If this is the case, it is natural to assume that expectations of different agents can also differ. We will make the further distinction that heterogeneity in expectations can be due to different initial beliefs or the use of different learning algorithms by the agents. Clearly, structurally homogenous economies can exhibit heterogenous expectations (and this possibility is permitted in some studies, see the references below), but different agents respond to expectations in the same way in economies with structural homogeneity.

While different approaches to adaptive learning exist, probably the largest concentration of research has used what is called the statistical or econometric learning.[3] In this approach the economic agents are assumed to use forecast functions that depend on some parameters and, at any moment of time, the economic decisions are made on the basis of expectations/forecasts obtained from these functions. The values of the parameters in these functions and expectations of the agents are adjusted over time as new data becomes available. Parameter updating is assumed to be done using standard econometric methods such as recursive least squares (RLS) estimation. A key issue of interest is whether this kind of adaptive learning behavior converges to a rational expectations equilibrium (REE) over time. If this is the case, then eventually the forecast functions of the agents are those associated with the REE.

In this paper our goal is to consider the stability of REE when both structural and expectational heterogeneity is present. The basic framework will be a multivariate linear model with two classes of agents. While the assumption of linearity is directly postulated for some models in the literature, we can also justify this assumption by observing that

[1](Evans and Honkapohja 2001) is a recent treatise on the subject. For overviews and surveys see e.g. (Evans and Honkapohja 1999), (Marimon 1997), (Sargent 1993) and (Sargent 1999).

[2]This terminology is introduced in Chapter 2 of (Evans and Honkapohja 2001).

[3]Other approaches to adaptive learning include the use of computational intelligence (see e.g. (Arifovic 1998)), models of discrete predictor choice (see e.g. (Brock and Hommes 1997) and (Brock and de Fontnouvelle 2000)) and eductive learning (see (Guesnerie 1999)). In addition, adaptive learning is usually a part of the so-called agent based models, see e.g. (LeBaron 2001). These alternative approaches rely heavily on simulation studies. Theoretical results are available for models of discrete predictor choice and eductive learning.

most applied studies are in any case based on linearization.[4] The restriction to two classes of agents in the main analysis is done only for simplicity of formulation, and we also state the stability conditions for economies with a finite number of different classes of agents. As already noted, heterogenous expectations can arise because of different initial beliefs or because the learning rules of the agents differ and we will analyze both possibilities.[5] We also take up the case in which one class of agents has continually RE while others are learning as this case occasionally appears in the literature.

The economy may be purely forward looking or it may also include lags of endogenous variables. Though we will work out the details in the forward looking framework, the main analytical steps for economies with lags will also be outlined. We will use the general stability conditions in three economic applications: Muth's market model, a model of monetary policy and Cagan's inflation model. Our analysis is focused on models where different agents need to forecast a common vector of aggregate variables, which is a very common setting in the literature. In other words, we will assume that information is symmetric between the agents. This is done for simplicity and brevity, though we conjecture that the approach can be generalized to models with informational asymmetries once the concept of equilibrium is suitably modified. Informational asymmetries are obviously a further source for heterogeneity in expectations.[6]

In the earlier literature, the bulk of the work on econometric learning has assumed homogeneity in both expectations and structure, though there exist several studies that permit heterogenous expectations in a homogenous structure, see e.g. (Bray and Savin 1986), (Evans and Honkapohja 1997), (Evans, Honkapohja, and Marimon 2001) and (Giannitsarou 2001). Heterogenous expectations are also present in some of the other approaches to adaptive learning cited in footnote 2 above. Structural heterogeneity is permitted for a class of models in (Marcet and Sargent 1989a). Expectations are heterogenous in the Marcet and Sargent setup, but this arises solely from informational differences as different agents are assumed to use versions of RLS estimation as their learning algorithms. Moreover, Marcet and Sargent do not provide explicit stability conditions in terms of the structural parameters of the economy.[7]

[4] We conjecture that it would be relatively straightforward to extend our analysis to learning of steady states, cycles and sunspots in the nonlinear setups considered in Part 4 of (Evans and Honkapohja 2001).

[5] In independent work (Giannitsarou 2001) considers similar forms of heterogeneity under the restrictive assumption of structural homogeneity, so that the economy depends only on the average expectations of the agents.

[6] For recent work see e.g. (Evans and Honkapohja 2001), Chapter 13 and (Honkapohja and Mitra 2002), Section 5. An early paper by (Marcet and Sargent 1989a) has a setup with informational asymmetries.

[7] They employ a restrictive version of the stochastic approximation methodology by using the so-called projection facility. Its use has been criticized especially in connection with heterogenous expectations and differential information, see (Grandmont and Laroque 1991), (Grandmont 1998) and (Moreno and Walker 1994). Ways to avoid a projection facility are discussed in (Evans and Honkapohja 1998) and Chapter 6 of (Evans and Honkapohja 2001).

2 The Framework

We consider a class of multivariate linear (or linearized) models where there are two types of agents (1 and 2) with different forecasts and with structural heterogeneity. As already noted, some economic models have a linear structure, but linearity also follows from the common practice of linearization or log-linearization of a nonlinear model. For brevity, we do not develop the details of the linearization around a non-stochastic steady state.

The formal model is given by

$$y_t = \alpha + A_1 \hat{E}_t^1 y_{t+1} + A_2 \hat{E}_t^2 y_{t+1} + B w_t, \quad (1)$$

$$w_t = F w_{t-1} + \varepsilon_t. \quad (2)$$

Here y_t is $n \times 1$ vector of endogenous variables and w_t is k dimensional vector of exogenous variables that is assumed to follow a stationary VAR, so that ε_t is white noise. F is a diagonal matrix with all eigenvalues inside the unit circle.[8] For simplicity, it is assumed that F is known to the agents (if not, it could be estimated). Let $\lim_{t\to\infty} E w_t w_t' = M_w$. As for the matrices, A_1 is $n \times n$, A_2 is $n \times n$ while B is $n \times k$.

We let $\hat{E}_t^i y_{t+1}, i = 1, 2$, denote the (in general non-rational) expectations by agent i of the endogenous variables in the economy. Expectations without "^" refer to RE. Naturally, some of the endogenous variables may not be of interest to an agent i and in this case the relevant entries in the matrix A_i would be zero.

A key feature of model (1) is that both agents' characteristics and forecasts differ. If either agents or forecasts are identical, so that $A_1 = A_2$ or $\hat{E}_t^1 y_{t+1} = \hat{E}_t^2 y_{t+1}$, the model can be aggregated. In the former case the evolution of y_t depends only on average expectations, which has been analyzed in the earlier literature. In the latter case only the aggregate characteristics $A_1 + A_2$ matter and the model becomes homogenous.

We will focus attention on the learnability of the fundamental or minimal state variable (MSV) solution to the class of models (1)-(2).[9] This REE takes the form

$$y_t = a + b w_t, \quad (3)$$

where the n vector a and $n \times k$ matrix b are to be computed in terms of the structural parameters of the model. We will show a bit later that the MSV solution is generically unique and it can be obtained by solving the following system of linear equations

$$a = \alpha + (A_1 + A_2) a$$

$$b = (A_1 + A_2) b F + B,$$

[8] Diagonality of F is usually without loss of generality since a non-diagonal matrix can very often be diagonalized. In that case the shocks w_t would be some linear transformations of the original fundamental shocks. Sometimes we will explicitly assume further that F is both diagonal and positive.

[9] As is well known, under certain conditions, known as indeterminacy of REE, there also exist other well behaved REE in forward looking models and these could also be studied for learnability. See e.g. (Evans and Honkapohja 2001), Part III for a discussion of the homogenous expectations case. The techniques developed in our paper can be extended to the study of learnability of the other types of REE under structural heterogeneity.

where the latter equation can be vectorized to yield a system of linear equations.

It should be noted that the framework is restrictive in that the model (1)-(2) is purely forward-looking. This is done solely to simplify the presentation of the theoretical results. In Section 9 we will extend the analysis to the case of lagged endogenous variables:

$$y_t = \alpha + A_1 \hat{E}_t^1 y_{t+1} + A_2 \hat{E}_t^2 y_{t+1} + D y_{t-1} + B w_t. \tag{4}$$

We also note that the corresponding static model

$$y_t = \alpha + A_1 \hat{E}_{t-1}^1 y_t + A_2 \hat{E}_{t-1}^2 y_t + B w_t, \tag{5}$$

can be analyzed in the same way and the formal results apply to this case. Indeed, one of our economic applications will fit the form (5).

In the extension to $S > 2$ classes of agents the model becomes

$$y_t = \alpha + \sum_{s=1}^{S} A_s \hat{E}_t^s y_{t+1} + B w_t, \tag{6}$$

$$w_t = F w_{t-1} + \varepsilon_t. \tag{7}$$

We will give the convergence conditions for model (6)-(7) at the end of Section 6.

We also note that it is possible to extend the stability results in this paper to models with proportions of agents of different types. In this setting the matrices in (1) would have the form $A_1 = \varkappa_1 \tilde{A}_1$ and $A_2 = \varkappa_2 \tilde{A}_2$ for $\varkappa_1, \varkappa_2 > 0, \varkappa_1 + \varkappa_2 = 1$. We do not provide any explicit results on this last case, since it does not arise in our economic applications. It would clearly be straightforward to work out the details.

2.1 Economic Applications

Here we outline three economic models that fit our general setup.

Example 1 (Market model with structural heterogeneity)[10] The demand function for a single good is assumed to be linear and downward sloping, that is

$$d_t = l - k p_t + \varepsilon_t.$$

Here k, l are positive parameters and ε_t is a demand shock that follows the AR(1) process

$$\varepsilon_t = f \varepsilon_{t-1} + v_t,$$

where v_t is white noise with variance σ_v^2 and $|f| < 1$.

It is assumed that there are L classes of suppliers with different linear supply functions that depend on expected market price due to a production lag. Formally,

$$s_t^i = h_i + n_i \hat{E}_{t-1}^i p_t, \; i = 1, \ldots, L.$$

[10]The classic analysis of this model under RE and homogenous supplies was presented by (Muth 1961). Adaptive learning in the (homogenous) Muth model was studied by (Bray and Savin 1986) and (Fourgeaud, Gourieroux, and Pradel 1986). The model is sometimes called the cobweb model.

where h_i, n_i are positive parameters and $\hat{E}_{t-1}^i p_t$ denotes the (in general non-rational) expectation of producer i about the market price. Expectations for period t are formed at the end of period $t-1$ before the demand shock ε_t is realized. We make the technical assumption that $fk^{-1}n_i + 1 > 0$ for all $i = 1, \dots, L$.

From market clearing $d_t = \sum_{i=1}^L s_t^i$ we obtain the reduced form

$$p_t = k^{-1}(l - \sum_{i=1}^L h_i) - \sum_{i=1}^L k^{-1} n_i \hat{E}_{t-1}^i p_t + k^{-1}\varepsilon_t, \tag{8}$$

which is of the form (5). The analysis of this model will be completed in Section 7.

Example 2 (A model of monetary policy). Recent studies of monetary policy are often based on a model with representative consumer, monopolistic competition in product market and stickiness in price setting.[11] This leads to a bivariate linearized model of the form

$$z_t = -\phi(i_t - \hat{E}_t^P \pi_{t+1}) + \hat{E}_t^P z_{t+1} + g_t, \tag{9}$$

$$\pi_t = \lambda z_t + \beta \hat{E}_t^P \pi_{t+1} + u_t, \tag{10}$$

where z_t is the "output gap" i.e. the difference between actual and potential output, π_t is the inflation rate, i.e. the proportional rate of change in the price level from $t-1$ to t and i_t is the nominal interest rate. $\hat{E}_t^P \pi_{t+1}$ and $\hat{E}_t^P z_{t+1}$ denote *private sector* expectations of inflation and output gap next period. All the parameters in (9) and (10) are positive. $0 < \beta < 1$ is the discount rate of the representative firm.

u_t and g_t denote observable shocks that follow first order autoregressive processes:

$$\begin{pmatrix} u_t \\ g_t \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} u_{t-1} \\ g_{t-1} \end{pmatrix} + \begin{pmatrix} \hat{u}_t \\ \hat{g}_t \end{pmatrix}, \tag{11}$$

where $0 < |\mu| < 1, 0 < |\rho| < 1$ and $\hat{g}_t \sim iid(0, \sigma_g^2), \hat{u}_t \sim iid(0, \sigma_u^2)$. g_t represents shocks to government purchases as well as shocks to potential GDP. u_t represents any cost push shocks to marginal costs other than those entering through z_t.

The model is complete once an interest rate rule by the central bank, such as

$$i_t = \chi_0 + \chi_\pi \hat{E}_t^{CB} \pi_{t+1} + \chi_z \hat{E}_t^{CB} z_{t+1} + \chi_g g_t + \chi_u u_t, \tag{12}$$

is postulated. This rule is forward looking, i.e. depends on forecasts $\hat{E}_t^{CB} \pi_{t+1}, \hat{E}_t^{CB} z_{t+1}$ of inflation and output gap by the central bank. χ_i are parameters set by the central bank and they indicate how the bank responds to the values of the endogenous and exogenous variables. Interest rate rules such as (12) can arise from implementing optimal discretionary monetary policies, nominal GDP targeting or as a postulated Taylor

[11] See (Clarida, Gali, and Gertler 1999) for a survey. The original nonlinear model and its linearization is given e.g. in (Woodford 1996). Price stickiness is modeled along the lines of (Calvo 1983). Learning issues for these models are studied e.g. in (Bullard and Mitra 2001) and (Evans and Honkapohja 2000).

rule, see (Evans and Honkapohja 2000), (Mitra 2001) and (Bullard and Mitra 2001), respectively.[12] Substituting (12) into (9) leads to a bivariate model of the form (1).

The above setting with private sector expectations and internal central bank forecasts very naturally involves heterogeneity in both expectations and economic structure. We will continue the analysis of this model in Section 8.1.

Example 3 (Cagan model with variable money supply)[13] Assume that there are two types of agents $s = 1, 2$ in the generation born at time t and having two-period life spans. They have utility functions $U^s = u(c_t^s) + \beta u(c_{t+1}^s)$ with the budget constraint $P_t(c_t^s - \omega_1^s) + \hat{E}_t^s P_{t+1}(c_{t+1}^s - \omega_2^s) \leq 0$, where ω_1^s, ω_2^s denotes the endowments of agent s and $\hat{E}_t^s P_{t+1}$ is the expectations of the next period's price of the good by agent s of generation t. Maximizing utility leads to the first order condition

$$(\hat{E}_t^s P_{t+1}/P_t) u'[\omega_1^s - (\hat{E}_t^s P_{t+1}/P_t)(c_{t+1}^s - \omega_2^s)] = \beta u'(c_{t+1}^s),$$

which can be log-linearized and centered around a non-stochastic steady state to yield demand functions for real balances that depend negatively on expected inflation.

Letting m_t denote the log of the money supply and p_t the log of the price level at time t, the money market equilibrium takes the form

$$m_t - p_t = -g_1(\hat{E}_t^1 p_{t+1} - p_t) - g_2(\hat{E}_t^2 p_{t+1} - p_t),$$

where g_i are positive parameters. Money supply is assumed to respond to lagged price level, so that

$$m_t = d p_{t-1} + e_t,$$

where e_t is an *iid* shock to money supply. On economic grounds, we assume that $|d| < 1$.

These equations can be solved for the reduced form

$$p_t = \omega p_{t-1} + \eta_1 \hat{E}_t^1 p_{t+1} + \eta_2 \hat{E}_t^2 p_{t+1} + v_t, \tag{13}$$

where $\omega = (1 + g_1 + g_2)^{-1} d$, $\eta_i = (1 + g_1 + g_2)^{-1} g_i$ and v_t is a linear function of the shock e_t. We will assume that v_t is $i.i.d$ with zero mean and variance σ_v^2. Observe that $|\omega| < 1$ given our assumptions. Model (13) is an example of the form (4) with lagged endogenous variable. We will return to this application in Section 9.2.

3 Heterogenous Forecasts and Expectational Stability

A mapping from the perceptions of the economic agents to the resulting temporary equilibrium of the economy has turned out to be the key relationship in the study of

[12] These papers consider the case where private and central bank forecasts are assumed to be identical.

[13] This model is due to (Cagan 1956). For analysis of the model (with homogenous agents) under learning, see e.g. Chapter 8 of (Evans and Honkapohja 2001).

convergence of adaptive learning dynamics. In this section we develop the form of this mapping in the framework with heterogenous expectations and structure and establish the uniqueness of the MSV equilibrium.

It has been observed for a wide variety of different models that convergence of learning to REE (under homogenous forecasts and learning) obtains if and only if the REE satisfies certain stability conditions, known as E-stability conditions. In this section we extend E-stability conditions for heterogenous forecasts. Later we show that the same conditions govern convergence under actual real time learning as long as the two agents use learning algorithms that are asymptotically identical in a sense defined later.

We now derive the (extended) E-stability conditions. We assume that the two types of agents have different forecast functions, though they take the same parametric form. During the learning dynamics the agents have different beliefs about the parameters they are estimating, and these beliefs are adjusted over time. For given values of the parameters of the forecast function of each agent i, called the perceived law of motion (PLM) of agent i, one computes the actual law of motion (ALM) implied by the structure of the economy. E-stability is then determined by the differential equation in which the PLM parameters adjust in the direction of the ALM parameter values.

Define the vector of state variables $z_t = (1, w_t)'$ and the matrix of parameters $\varphi_i' = (a_i, b_i)$ with a_i being an n dimensional vector and b_i being an $n \times k$ matrix. Formally, we assume that the two agents have PLMs

$$y_t = a_1 + b_1 w_t = \varphi_1' z_t, \quad (14)$$
$$y_t = a_2 + b_2 w_t = \varphi_2' z_t, \quad (15)$$

with corresponding forecast functions

$$\hat{E}_t^1 y_{t+1} = a_1 + b_1 F w_t, \quad (16)$$
$$\hat{E}_t^2 y_{t+1} = a_2 + b_2 F w_t. \quad (17)$$

Note that the PLMs have the same form as the MSV solution (3), but in general a_i, b_i are not at their RE values. Inserting these forecasts into the model (1), one obtains the ALM

$$\begin{aligned} y_t &= \alpha + A_1 a_1 + A_2 a_2 + [(A_1 b_1 + A_2 b_2)F + B] w_t \\ &= [\alpha + A_1 a_1 + A_2 a_2, (A_1 b_1 + A_2 b_2)F + B] \begin{bmatrix} 1 \\ w_t \end{bmatrix} \\ &= T(\varphi_1', \varphi_2') z_t. \quad (18) \end{aligned}$$

We look at stability of the REE where the two agents have homogenous forecast functions, i.e., when $a_1 = a_2 = a$ and $b_1 = b_2 = b$. Before obtaining the E-stability conditions, we first show that this symmetric MSV solution is unique.

Proposition 1 *There exists a unique, symmetric equilibrium of the model (1)-(2) if the matrices $I_n - A_1 - A_2$ and $I_{nk} - F' \otimes (A_1 + A_2)$ are invertible.*

Here and in the rest of the paper I_m denotes the $m-$dimensional identity matrix.

Proof. The $T-$map, given by (18) and written out explicitly, is

$$\begin{aligned} a_1 &\to \alpha + A_1 a_1 + A_2 a_2, && (19)\\ a_2 &\to \alpha + A_1 a_1 + A_2 a_2, && (20)\\ b_1 &\to (A_1 b_1 + A_2 b_2)F + B, && (21)\\ b_2 &\to (A_1 b_1 + A_2 b_2)F + B. && (22) \end{aligned}$$

It is clear that equations (19)-(20) are symmetric in $(a_1,\ a_2)$ and equations (21)-(22) are symmetric in $(b_1,\ b_2)$, respectively. Thus $a_1 = a_2 = a$ and $b_1 = b_2 = b$ provided there exists a unique solution. For the a_1, a_2 system the solution is evidently unique if $I - A_1 - A_2$ is invertible.

The b_1, b_2 system needs to be vectorized[14]

$$vecb_1 = (F' \otimes A_1)vecb_1 + (F' \otimes A_2)vecb_2 + vecB,$$
$$vecb_2 = (F' \otimes A_1)vecb_1 + (F' \otimes A_2)vecb_2 + vecB.$$

The vectorized system can be rewritten as

$$\begin{pmatrix} I_{nk} - F' \otimes A_1 & -F' \otimes A_2 \\ -F' \otimes A_1 & I_{nk} - F' \otimes A_2 \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} = \begin{pmatrix} vecB \\ vecB \end{pmatrix},$$

which has a unique solution provided the left hand matrix is invertible. The determinant of this matrix is easily seen to be non-zero if and only if the matrix $I - F' \otimes (A_1 + A_2)$ is invertible. ■

We next formulate the differential equation defining E-stability:

$$\begin{aligned} \dot{a}_1 &= \alpha + (A_1 - I)a_1 + A_2 a_2, && (23)\\ \dot{b}_1 &= A_1 b_1 F - b_1 + A_2 b_2 F + B, && (24)\\ \dot{a}_2 &= \alpha + A_1 a_1 + (A_2 - I)a_2, && (25)\\ \dot{b}_2 &= A_1 b_1 F + A_2 b_2 F - b_2 + B. && (26) \end{aligned}$$

The system involving $\dot{a}_1, \dot{a}_2$ is independent from the system for $\dot{b}_1$ and $\dot{b}_2$, and it can be written as

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} A_1 - I_n & A_2 \\ A_1 & A_2 - I_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}. \tag{27}$$

The system for $\dot{b}_1, \dot{b}_2$ needs to be vectorized and it can be written as

$$\begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} = \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} + \begin{pmatrix} vecB \\ vecB \end{pmatrix}. \tag{28}$$

We can now prove the following proposition:

[14] Here F' denotes the transpose. As F is assumed to be diagonal, this notation is not really necessary. We have kept the transposes as the same formulae then hold for a nonsymmetric F matrix as well.

Proposition 2 *Consider the model (1)-(2) with the PLMs of the agents (14)-(15), their forecasts (16)-(17) and the ALM (18). The E-stability conditions extended for heterogenous expectations are the same as when the agents have homogenous forecasts. The symmetric equilibrium is E-stable if and only if the matrices* A_1+A_2-I *and* $F'\otimes(A_1+A_2)-I$ *have eigenvalues with negative real parts.*[15]

Proof. The differential equations defining E-stability (23)-(26) are locally stable at the symmetric equilibrium if and only if the eigenvalues of the matrices on the right hand sides of (27) and (28) have negative real parts. To shorten notation, define

$$A \equiv \begin{pmatrix} A_1 - I_n & A_2 \\ A_1 & A_2 - I_n \end{pmatrix}, \tag{29}$$

$$F_1 \equiv \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix}. \tag{30}$$

The determinant for computing the eigenvalues of (29), $|A - mI_{2n}|$, may be simplified as follows:

$$\begin{aligned} |A - mI_{2n}| &= \begin{vmatrix} A_1 - I_n(1+m) & A_2 \\ A_1 & A_2 - I_n(1+m) \end{vmatrix} \\ &= \begin{vmatrix} -I_n(1+m) & I_n(1+m) \\ A_1 & A_2 - I_n(1+m) \end{vmatrix} \\ &= \begin{vmatrix} -I_n(1+m) & 0 \\ A_1 & A_1 + A_2 - I_n(1+m) \end{vmatrix} \\ &= (-(1+m))^n \, |A_1 + A_2 - I_n(1+m)| . \end{aligned}$$

The computation shows that A has n eigenvalues equal to -1 and the remaining eigenvalues are those of $A_1 + A_2 - I_n$. Hence, A has eigenvalues with negative real parts if and only if $A_1 + A_2 - I_n$ has the same property.

Analogously, the determinant for computing the eigenvalue of the coefficient matrix F_1 in (30) can be written as (after subtracting the second row from the first)

$$\begin{aligned} &\begin{vmatrix} -(1+m)I_{nk} & (1+m)I_{nk} \\ F' \otimes A_1 & F' \otimes A_2 - (1+m)I_{nk} \end{vmatrix} \\ &= \{-(1+m)\}^{nk} \, |F' \otimes (A_1 + A_2) - (1+m)I_{nk}| . \end{aligned}$$

so that F_1 has nk eigenvalues equal to -1 and the rest are the eigenvalues of $F' \otimes (A_1 + A_2) - I_{nk}$. Consequently, F_1 will have eigenvalues with negative real parts if and only if $F' \otimes (A_1 + A_2) - I_{nk}$ has so.

Finally, the result follows since when $\hat{E}_t^1 y_{t+1} = \hat{E}_t^2 y_{t+1} = \hat{E}_t y_{t+1}$, the matrix in front of the common expectations $\hat{E}_t y_{t+1}$ in (1) becomes $A_1 + A_2$, which is the homogenous case. ■

[15] Throughout the paper we ignore the non-generic cases where one or more relevant eigenvalues has a zero real part.

We remark that if F is a positive, diagonal matrix, the E-stability conditions simplify to condition that the eigenvalues of $A_1 + A_2 - I_n$ have negative real parts.

The next section will demonstrate that the stability of the system under certain forms of heterogenous learning rules obtains if and only if the above E-stability conditions are satisfied. In actual real time learning the two agents use versions of (generalized) recursive least squares in their updating of estimates of parameters which are relevant to their forecasting. However, the learning rules can start with different initial beliefs about the parameters so that they differ along the path. The E-stability conditions, therefore, govern convergence to REE even when we allow this (limited) form of heterogeneity in learning. The analysis thus shows that the stability conditions for the homogenous case are not as restrictive as they may seem - homogeneity in forecasting and learning is a good first approximation.

4 RLS Learning with Different Initial Beliefs

We now consider the learning by agents in real time when they use versions of (generalized) recursive least squares in the updating of parameter estimates relevant to their forecasting. Assume that the perceived laws of motion (PLM) of agents 1 and 2 are, respectively,

$$y_t = a_{1,t} + b_{1,t} w_t = \varphi'_{1,t} z_t, \quad (31)$$
$$y_t = a_{2,t} + b_{2,t} w_t = \varphi'_{2,t} z_t, \quad (32)$$

where we note that the estimates of parameters, $\varphi'_{1,t}$ and $\varphi'_{2,t}$, are now time dependent.

The corresponding forecast functions are

$$\hat{E}_t^1 y_{t+1} = a_{1,t} + b_{1,t} F w_t, \quad (33)$$
$$\hat{E}_t^2 y_{t+1} = a_{2,t} + b_{2,t} F w_t. \quad (34)$$

In this formulation the parameter estimates are assumed to depend on data up to $t-1$, but current observation on exogenous variables are allowed to be used in the forecasts. (This is typically done in the learning literature.) Using these forecasts, the ALM of y_t is then given (as before) by

$$y_t = T(\varphi'_{1,t}, \varphi'_{2,t}) z_t, \quad (35)$$

where T is the map appearing in (18).

We assume in this section that both types of agents use versions of recursive least squares (RLS) but they can have different initial beliefs of the parameter estimates. More specifically, agents 1 and 2 use the following learning algorithms

$$\varphi_{1,t} = \varphi_{1,t-1} + \gamma_{1,t} R_{1,t}^{-1} z_{t-1}(y_{t-1} - \varphi'_{1,t-1} z_{t-1})', \quad (36)$$
$$R_{1,t} = R_{1,t-1} + \gamma_{1,t}(z_{t-1} z'_{t-1} - R_{1,t-1}), \quad (37)$$
$$\varphi_{2,t} = \varphi_{2,t-1} + \gamma_{2,t} R_{2,t}^{-1} z_{t-1}(y_{t-1} - \varphi'_{1,t-1} z_{t-1})', \quad (38)$$
$$R_{2,t} = R_{2,t-1} + \gamma_{2,t}(z_{t-1} z'_{t-1} - R_{2,t-1}). \quad (39)$$

Different initial beliefs can be accommodated by different initial conditions for the dynamics. The gain parameters $\gamma_{i,t} > 0$ indicate responsiveness of the change in parameter estimates to forecast errors and new data. They satisfy $\lim_{t\to\infty} \gamma_{i,t} = 0$ and $\sum \gamma_{i,t} = \infty$. RLS is the case where $\gamma_{it} = t^{-1}$. We allow for $\gamma_{1,t} \neq \gamma_{2,t}$ for the gain parameters of the learning rules and make the following assumption:

Assumption A: There exists a non-increasing positive sequence γ_t with properties:
(i) $\gamma_{i,t} \leq K_i \gamma_t$ for some constant $K_i > 0$,
(ii) $\sum \gamma_t = \infty$ and $\sum \gamma_t^p < \infty$ for some $p \geq 2$, and
(ii) $\limsup(1/\gamma_{t+1} - 1/\gamma_t) < \infty$.

We remark that these conditions on γ_t are commonly assumed in the literature.[16] Assumption A can allow various weighting schemes for data in later periods relative to early ones, see e.g. (Ljung and Söderström 1983) and (Marcet and Sargent 1989b).

However, we assume that asymptotically the gain sequences converge at the same rate, that is,

Condition 1: $\gamma_{1,t}\gamma_t^{-1} \to \delta$ and $\gamma_{2,t}\gamma_t^{-1} \to \delta$ as $t \to \infty$.

With these assumptions we have the following result:

Theorem 3 *Consider the model (1)-(2) with the PLMs (31)-(32), the forecasts (33)-(34), the learning algorithms (36)-(39), and the ALM (35). Assume furthermore that Assumption A and Condition 1 hold. If the symmetric equilibrium is E-stable, then the learning algorithms converge to this equilibrium.*[17]

Proof. This is a special case of Theorem 4, see Section 5. ■

We also note that under some (mild) regularity conditions, the RLS algorithm will converge to an E-unstable symmetric (MSV) solution with probability zero, see (Evans and Honkapohja 2001) for the details. The conclusion of this section is thus that convergence with some forms of heterogeneity in learning continues to be governed by the standard E-stability conditions.

As a technical remark we note that this kind of convergence and non-convergence results are formally established by deriving the so-called associated ordinary differential equation (ODE) of the stochastic recursive algorithm governing convergence and non-convergence of learning.[18] (Moreover, the ODE defining E-stability is directly linked

[16] We note that one can assume $K_i \leq 1$ without loss of generality. If γ_t satisfies Assumption A for $K_i > 1$, then one can construct another sequence $\tilde{\gamma}_t$ satisfying assumption A with the constant $K_i \leq 1, \forall i$.

[17] We have not dwelled into the precise notion of convergence that obtains in these kinds of models. The assumption of a "projection facility", which was used in the early literature, can be relaxed at the cost of the strength in the sense of convergence. See (Evans and Honkapohja 2001), Chapter 6, Sections 3 and 4 and (Evans and Honkapohja 1998) for detailed discussion.

[18] See e.g. Chapters 6 and 7 of (Evans and Honkapohja 2001) or (Evans and Honkapohja 1998).

to the associated ODE of the algorithm.) The just noted relationship between stability or instability in the associated ODE and the convergence or non-convergence of the algorithm also applies to other settings below. Thus, below we only state stability and convergence results, but it should be kept in mind that corresponding instability/nonconvergence results also exist.

5 RLS Learning with Different Gain Sequences

We now analyze asymptotic heterogeneity in gain sequences in the learning algorithms. The formulation includes and generalizes the heterogeneities in the weights considered in the preceding section. Our formulation includes inertia in updating of forecast rules and even independent random fluctuations in adaption speeds.[19] Otherwise, we assume that the algorithms of the two agents are of the RLS type, i.e. they are given by (36)-(39).

The individual gain sequences are assumed to satisfy:

Condition 2: $\gamma_{i,t} = \hat{\gamma}_{i,t}\xi_{i,t}$, where the random gains $\hat{\gamma}_{i,t}$ are positive, independent of past information and across agents, and $\xi_{i,t}$ is a Bernoulli random variable equal to 0 with probability $\rho_{i,t} \in [0,1)$ and equal to 1 with probability $1-\rho_{i,t}$. $\xi_{i,t}$ is independent of past information and $\hat{\gamma}_{i,t}$. In addition, $\lim_{t\to\infty} E(\xi_{i,t}\hat{\gamma}_{i,t}/\gamma_t) = \delta_i > 0$, where the deterministic sequence γ_t satisfies Assumption A in the preceding section.

This condition allows for significant amounts of heterogeneity, including both randomness and inertia indicated by $\hat{\gamma}_{i,t}$ and $\xi_{i,t}$, respectively, in the adaption speeds of the different agents. A similar formulation of heterogeneity in learning was used in (Evans, Honkapohja, and Marimon 2001). Effectively, the above condition means that the gain sequences of the two agents converge (in mean) at different rates even asymptotically.

Formally, the dynamics continues to be given by the system (36)-(39), except that the gain sequences now have different properties. The technical analysis of the algorithm is outlined in Appendix A.1 and in this case the associated ODE is

$$\begin{aligned} d\varphi_1/d\tau &= \delta_1 S_1^{-1} M_z (T(\varphi_1', \varphi_2')' - \varphi_1), \\ dS_1/d\tau &= \delta_1 (M_z - S_1), \\ d\varphi_2/d\tau &= \delta_2 S_2^{-1} M_z (T(\varphi_1', \varphi_2')' - \varphi_2), \\ dS_2/d\tau &= \delta_2 (M_z - S_2). \end{aligned} \quad (40)$$

where the T map continues to be given by (18) and

$$lim_{t\to\infty} E z_{t-1} z_{t-1}' = M_z = \begin{pmatrix} 1 & 0 \\ 0 & M_w \end{pmatrix}. \quad (41)$$

[19] Inertia in the formation of expectations is observed in experimental data, see for instance (Marimon and Sunder 1993) and (Evans, Honkapohja, and Marimon 2001).

Note that M_w is a diagonal, positive definite matrix, since F in (2) was assumed to be diagonal. Since $S_1 \to M_z$ and $S_2 \to M_z$ stability is governed by the smaller differential equation

$$
\begin{aligned}
d\varphi_1/d\tau &= \delta_1(T(\varphi_1', \varphi_2')' - \varphi_1), \qquad (42)\\
d\varphi_2/d\tau &= \delta_2(T(\varphi_1', \varphi_2')' - \varphi_2). \qquad (43)
\end{aligned}
$$

We first note that if $\delta_1 = \delta_2$ then the stability conditions obtained from (42)-(43) would be identical to the E-stability conditions, which proves Theorem 3.

Returning to the general case and rearranging (42)-(43), we get

$$
\begin{aligned}
\dot{\varphi}_1' &= [\delta_1(\alpha + A_1 a_1 + A_2 a_2 - a_1), \delta_1(A_1 b_1 F + A_2 b_2 F + B - b_1)],\\
\dot{\varphi}_2' &= [\delta_2(\alpha + A_1 a_1 + A_2 a_2 - a_2), \delta_2(A_1 b_1 F + A_2 b_2 F + B - b_2)],
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{a}_1 &= \delta_1(A_1 - I)a_1 + \delta_1 A_2 a_2 + \delta_1 \alpha,\\
\dot{b}_1 &= \delta_1 A_1 b_1 F + \delta_1 A_2 b_2 F + \delta_1 B - \delta_1 b_1,\\
\dot{a}_2 &= \delta_2 A_1 a_1 + \delta_2(A_2 - I)a_2 + \delta_2 \alpha,\\
\dot{b}_2 &= \delta_2 A_1 b_1 F + \delta_2 A_2 b_2 F + \delta_2 B - \delta_2 b_2.
\end{aligned}
$$

We look at stability of the symmetric equilibrium where $a_1 = a_2 = a$ and $b_1 = b_2 = b$. The system for $\dot{a}_1$ and $\dot{a}_2$ can be written as

$$
\begin{aligned}
\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} &= \begin{pmatrix} \delta_1(A_1 - I_n) & \delta_1 A_2 \\ \delta_2 A_1 & \delta_2(A_2 - I_n) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \delta_1 \alpha \\ \delta_2 \alpha \end{pmatrix}\\
&= \begin{pmatrix} \delta_1 I_n & 0 \\ 0 & \delta_2 I_n \end{pmatrix} \begin{pmatrix} A_1 - I_n & A_2 \\ A_1 & A_2 - I_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \delta_1 \alpha \\ \delta_2 \alpha \end{pmatrix} \qquad (44)\\
&\equiv D_1 A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \delta_1 \alpha \\ \delta_2 \alpha \end{pmatrix}.
\end{aligned}
$$

where D_1 is the diagonal matrix appearing in (44) and A is the matrix in (29). For stability we need the matrix $D_1 A$ to have eigenvalues with negative real parts.

The system for $\dot{b}_1$ and $\dot{b}_2$ needs to be vectorized as before to yield (ignoring constant terms)

$$
\begin{aligned}
\begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} &= \begin{pmatrix} F' \otimes \delta_1 A_1 - \delta_1 I_{nk} & F' \otimes \delta_1 A_2 \\ F' \otimes \delta_2 A_1 & F' \otimes \delta_2 A_2 - \delta_2 I_{nk}) \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix}\\
&= \begin{pmatrix} \delta_1 I_{nk} & 0 \\ 0 & \delta_2 I_{nk} \end{pmatrix} \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix}\\
&\equiv D_2 F_1 \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} \qquad (45)
\end{aligned}
$$

where F_1 is defined as in (30). The eigenvalues of $D_2 F_1$ must have negative real parts for stability of the above system. We have thus obtained the following result:

Theorem 4 *Consider the model (1)-(2) under modified recursive least squares (RLS) learning, given by (36)-(39), and assume Condition 2. If the matrices D_1A and D_2F_1, given in (44) and (45), have eigenvalues with negative real parts, then the learning algorithms converge to the symmetric equilibrium.*

This theorem shows that the stability conditions are in general affected by δ_1 and δ_2. In some cases it is, however, possible to provide sufficient conditions for stability that do not depend on δ_1 and δ_2. For this we need the notion of D-stability.[20] A matrix A is said to be D-stable if the matrix DA has all eigenvalues with negative real parts for any positive diagonal matrix D. Using this one has the following corollary:

Corollary 5 *Consider the model (1)-(2) under RLS learning, given by (36)-(39), and assume Condition 2. If the matrices A and F_1, given in (29) and (30), are D-stable, then the learning algorithms converge to the symmetric equilibrium.*

The proof of this corollary is immediate from (44)-(45). Evidently, the requirement of D-stability is restrictive and, indeed, the monetary model of Example 2 does not satisfy this definition. However, the matrices in Example 1 do satisfy D-stability, as will be shown later in Section 7.

6 RLS Learning and SG Learning

We now consider the case when the agents are using quite different algorithms in their updating schemes. The broad aim is to consider settings where one class of agents is using a learning algorithm that is either more or less sophisticated than the algorithm used by the other class of agents. Specifically, we assume that there are two possible types of learning algorithms, the RLS and the stochastic gradient (SG) algorithms that the agents might use. (The RLS algorithm is more commonly employed than SG in the literature.)

The SG algorithm is computationally much simpler than the RLS algorithm; however the latter is more efficient from an econometric viewpoint since it uses information on the second moments of the variables. For parameter estimation of fixed exogenous stochastic processes, both the RLS and SG algorithms yield consistent estimates of parameters but the RLS, in addition, possesses some optimality properties. For instance, if the underlying shock process is *iid* normal, then the RLS estimator is minimum variance unbiased.[21]

Formally, we assume that agent 1 updates the parameter estimates using an RLS algorithm while agent 2 updates using a stochastic gradient (SG) type algorithm. The

[20] This concept has been used earlier in the literature on Walrasian tatonnement dynamics, see (Arrow and McManus 1958), (Enthoven and Arrow 1956), and (Johnson 1974).

[21] See (Evans and Honkapohja 2001), Section 3.5 for a discussion and references to SG learning. We note that these properties refer to the usual statistical analysis that involves parameter estimation for exogenous processes.

SG algorithm is simpler than RLS as it does not make use of the matrix of second moments, see Chapter 3 of (Evans and Honkapohja 2001) for further discussion.

For agent 1 the algorithm is given by

$$\begin{aligned} \varphi_{1,t} &= \varphi_{1,t-1} + \gamma_t(\gamma_{1,t}\gamma_t^{-1})R_t^{-1}z_{t-1}(y_{t-1} - \varphi'_{1,t-1}z_{t-1})', && (46) \\ R_t &= R_{t-1} + \gamma_t(\gamma_{1,t}\gamma_t^{-1})(z_{t-1}z'_{t-1} - R_{t-1}), && (47) \end{aligned}$$

while for agent 2 it is given by

$$\varphi_{2,t} = \varphi_{2,t-1} + \gamma_t(\gamma_{2,t}\gamma_t^{-1})z_{t-1}(y_{t-1} - \varphi'_{1,t-1}z_{t-1})'. \tag{48}$$

In addition, we assume

Condition 3: $\lim_{t\to\infty}(\gamma_{1,t}\gamma_t^{-1}) \to 1$ and $\lim_{t\to\infty}(\gamma_{2,t}\gamma_t^{-1}) \to 1$.

The technical analysis of the algorithm (46), (47) and (48) is given in Appendix A.1. In this case the associated ODE governing convergence of the algorithm boils down to

$$\begin{aligned} d\varphi_1/d\tau &= S^{-1}M_z(T(\varphi'_1, \varphi'_2)' - \varphi_1), \\ dS/d\tau &= M_z - S, \\ d\varphi_2/d\tau &= M_z(T(\varphi'_1, \varphi'_2)' - \varphi_2). \end{aligned}$$

Since the second set of equations is globally stable with $S \to M_z$ from any starting point, stability is determined entirely by the smaller dimensional system

$$\begin{aligned} d\varphi_1/d\tau &= (T(\varphi'_1, \varphi'_2)' - \varphi_1), \\ d\varphi_2/d\tau &= M_z(T(\varphi'_1, \varphi'_2)' - \varphi_2). \end{aligned}$$

This immediately shows that the E-stability conditions are no longer sufficient for convergence of learning dynamics although they continue to be necessary. In particular, the moment matrix M_z affects the stability conditions.

By usual arguments,

$$\begin{aligned} \dot{\varphi}'_1 &= [\alpha + A_1a_1 + A_2a_2 - a_1, (A_1b_1 + A_2b_2)F + B - b_1], && (49) \\ \dot{\varphi}'_2 &= [\alpha + A_1a_1 + A_2a_2 - a_2, (A_1b_1 + A_2b_2)F + B - b_2]M_z \\ &= [\alpha + A_1a_1 + A_2a_2 - a_2, \{(A_1b_1 + A_2b_2)F + B - b_2\}M_w] \end{aligned}$$

as well as

$$\begin{aligned} \dot{a}_1 &= \alpha + (A_1 - I_n)a_1 + A_2a_2, \\ \dot{b}_1 &= A_1b_1F - b_1 + A_2b_2F + B, \\ \dot{a}_2 &= \alpha + A_1a_1 + (A_2 - I_n)a_2, \\ \dot{b}_2 &= A_1b_1FM_w + A_2b_2FM_w - b_2M_w + BM_w. \end{aligned}$$

As before, we consider stability of the symmetric equilibrium. The system for $\dot{a}_1$ and $\dot{a}_2$ is the same as (27). This system is, therefore, stable if the eigenvalues of $A_1 + A_2$ have real parts less than 1. The system for $\dot{b}_1$ and $\dot{b}_2$ needs to be vectorized, so

$$\begin{aligned} vec\dot{b}_1 &= (F' \otimes A_1 - I_{nk})vecb_1 + (F' \otimes A_2)vecb_2 + vecB, \\ vec\dot{b}_2 &= (M'_w F' \otimes A_1)vecb_1 + (M'_w F' \otimes A_2 - M'_w \otimes I_n))vecb_2 + (M'_w \otimes I_n)vecB, \end{aligned}$$

or ignoring constant terms

$$\begin{aligned} \begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} &= \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ M'_w F' \otimes A_1 & M'_w F' \otimes A_2 - M'_w \otimes I_n \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} \\ &= \begin{pmatrix} I_{nk} & 0 \\ 0 & M_w \otimes I_n \end{pmatrix} \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} \\ &\equiv D_w F_1 \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix}, \end{aligned} \tag{50}$$

where D_w is the diagonal matrix in the second line of (50). We can then prove the following theorem:

Theorem 6 *Consider the model (1)-(2) where agent* 1 *uses recursive least squares (RLS) learning given by (46)-(47) and agent* 2 *uses the stochastic gradient algorithm (48). Assume, furthermore, Condition 3. If the matrices A and $D_w F_1$ have eigenvalues with negative real parts, then the learning dynamics converges to the symmetric equilibrium.*

We can also obtain a result analogous to Corollary 5 in Section 5. Since M_w is a diagonal, positive definite matrix, we have the analogy of Corollary 5:

Corollary 7 *Consider the model (1)-(2) where agent* 1 *uses recursive least squares (RLS) learning given by (46)-(47), agent* 2 *uses the stochastic gradient algorithm (48), and assume Condition 3. If A is stable (i.e. has eigenvalues with negative real parts) and F_1 is $D-$stable, then the learning algorithms converge to the symmetric equilibrium.*[22]

A common theme emerges from Corollaries 5 and 7. If both A and F_1 are $D-$stable, then the learning rules converge locally to the symmetric equilibrium irrespective of whether they are characterized by differential gains asymptotically, as in Section 5, or are of different types as in this section.

Finally, we note some extensions of the results to economies with more than two classes of agents and to global convergence of learning.

[22] We note that that if the matrix M_w is not diagonal, then F_1 would need to be $S-$stable, see (Arrow and McManus 1958) for the definition of $S-$stability.

Remarks on the model with $S > 2$ classes of agents: Consider the model (6)-(7) with S classes of agents. The E-stability condition is that the eigenvalues of the matrices

$$\sum_{s=1}^{S} A_s - I_n \text{ and } F' \otimes \sum_{s=1}^{S} A_s - I_{nk}$$

have negative real parts. This is also the convergence condition in the case of heterogenous beliefs but identical learning rules. If agents' learning rules have different gain sequences $\gamma_{s,t}$, the stability condition is that the matrices

$$D_1 A \equiv \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix} \begin{pmatrix} A_1 - I_n & \cdots & A_S \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_S - I_n \end{pmatrix} \text{ and } \quad (51)$$

$$D_2 F_1 \equiv \begin{pmatrix} \delta_1 I_{nk} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_{nk} \end{pmatrix} \begin{pmatrix} F' \otimes A_1 - I_{nk} & \cdots & F' \otimes A_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 & \cdots & F' \otimes A_S - I_{nk} \end{pmatrix} \quad (52)$$

have eigenvalues with negative real parts, where $\delta_s = \lim_{t\to\infty} E(\gamma_{s,t}/\gamma_t)$. D-stability of A and F_1 continues to be a sufficient condition for stability. In the case where some agents use RLS rules and others SG rules the stability condition is that the real parts of the eigenvalues of the matrices

$$\sum_{s=1}^{S} A_s - I_n \text{ and}$$

$$QF_1 \equiv \begin{pmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_S \end{pmatrix} \begin{pmatrix} F' \otimes A_1 - I_{nk} & \cdots & F' \otimes A_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 & \cdots & F' \otimes A_S - I_{nk} \end{pmatrix}$$

are negative, where $Q_s = I_{nk}$ or $M_w \otimes I_n$ if agent s is using RLS or SG, respectively. Stability of matrix A and D-stability of matrix F_1 are a sufficient condition for convergence in this case.

Remark on global vs. local convergence: In view of the linearity of the framework (1)-(2) it is evident that in Theorems 3, 4 and 6 convergence to the MSV REE is in fact global.[23] However, it should be borne in mind that in several applications the model is in fact a linearization around a steady state, and the study of learning is necessarily local in such settings. We have thus not specified in these theorems whether convergence is local or global.

[23] The conditions for global convergence of the resulting algorithms can be easily verified, see Section 7, Chapter 6 of (Evans and Honkapohja 2001).

It should be recalled that the model (1)-(2) can sometimes have other REE besides the MSV solutions that we have considered here. (The static model (6)-(7) does not have other REE.) For such equilibria, the forecast functions of the agents have additional parameters, so that dynamics do not operate in the same parameter space as here. It is also possible to inquire whether the MSV equilibrium remains stable under learning if such extra parameters are permitted (so that they would converge to zero over time). This issue has been considered using the concept of strong E-stability in frameworks with structural and expectational homogeneity, see e.g. Chapter 9 of (Evans and Honkapohja 2001) for discussion and references.

7 Application to Market Model

We now apply our results to the market model in Example 1. Note that the market model is of the form (5) with $A_s = -k^{-1}n_s$, $s = 1, \ldots, L$.

Assume that the suppliers have PLMs of the form

$$\hat{E}^s_{t-1}p_t = a_s + b_s\varepsilon_{t-1}.$$

Substituting these into the ALM (8) we get the reduced form

$$\begin{aligned} p_t &= k^{-1}(l - \sum_{s=1}^{L} h_s) - \sum_{s=1}^{L} k^{-1}n_s(a_s + b_s\varepsilon_{t-1}) + k^{-1}\varepsilon_t \\ &= k^{-1}(l - \sum_{s=1}^{L} h_s - \sum_{s=1}^{L} n_s a_s) - k^{-1}(\sum_{s=1}^{L} n_s b_s - f)\varepsilon_{t-1} + k^{-1}v_t. \end{aligned}$$

The implied forecasts are $(i = 1, 2)$

$$\hat{E}^i_{t-1}p_t = k^{-1}(l - \sum_{s=1}^{L} h_s - \sum_{s=1}^{L} n_s a_s) - k^{-1}(\sum_{s=1}^{L} n_s b_s - f)\varepsilon_{t-1}$$

and the T map is then

$$\begin{aligned} a_i &\to k^{-1}(l - \sum_{s=1}^{L} h_s - \sum_{s=1}^{L} n_s a_s), \\ b_i &\to -k^{-1}(\sum_{s=1}^{L} n_s b_s - f). \end{aligned}$$

The fixed points of the T map above give us the REE solution and it is easy to show that the symmetric solution (where $a_1 = \ldots = a_L \equiv \bar{a}$ and $b_1 = \ldots = b_L = \bar{b}$) is unique.

7.1 E-Stability of the REE

We now consider E-stability of this symmetric equilibrium. Dropping constant terms, for E-stability we can consider the differential equations

$$\frac{d}{d\tau}\begin{pmatrix} a_1 \\ \vdots \\ a_L \end{pmatrix} = -k^{-1}\begin{pmatrix} n_1 & \cdots & n_L \\ \vdots & \ddots & \vdots \\ n_1 & \cdots & n_L \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_L \end{pmatrix} - \begin{pmatrix} a_1 \\ \vdots \\ a_L \end{pmatrix}, \tag{53}$$

$$\frac{d}{d\tau}\begin{pmatrix} b_1 \\ \vdots \\ b_L \end{pmatrix} = -k^{-1}\begin{pmatrix} n_1 & \cdots & n_L \\ \vdots & \ddots & \vdots \\ n_1 & \cdots & n_L \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_L \end{pmatrix} - \begin{pmatrix} b_1 \\ \vdots \\ b_L \end{pmatrix}. \tag{54}$$

The eigenvalues of

$$-k^{-1}\begin{pmatrix} n_1 & \cdots & n_L \\ \vdots & \ddots & \vdots \\ n_1 & \cdots & n_L \end{pmatrix}$$

are obviously 0 and $-k^{-1}(\sum_{s=1}^{L} n_s)$. This proves the following result.

Proposition 8 *The symmetric equilibrium of the market model (8) under heterogenous forecasts is E-stable.*

7.2 Heterogenous Learning in the Market Model

Consider now the case when the suppliers have algorithms of the form (36)-(39) with the gain sequences satisfying Condition 2, i.e. they have differential gains asymptotically. The key matrices (51) and (52) for stability in this case are

$$\begin{pmatrix} -\delta_1(k^{-1}n_1+1) & -\delta_1 k^{-1}n_2 & \cdots & -\delta_1 k^{-1}n_L \\ -\delta_2 k^{-1}n_1 & -\delta_2(k^{-1}n_2+1) & \cdots & -\delta_1 k^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_2 k^{-1}n_1 & -\delta_1 k^{-1}n_2 & \cdots & -\delta_I(k^{-1}n_L+1) \end{pmatrix} \tag{55}$$

$$= \begin{pmatrix} \delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S \end{pmatrix}\begin{pmatrix} -(k^{-1}n_1+1) & -k^{-1}n_2 & \cdots & -k^{-1}n_L \\ -k^{-1}n_1 & -(k^{-1}n_2+1) & \cdots & -k^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -k^{-1}n_1 & -k^{-1}n_2 & \cdots & -(k^{-1}n_L+1) \end{pmatrix}$$

and

$$\begin{pmatrix} -\delta_1(fk^{-1}n_1+1) & -\delta_1 fk^{-1}n_2 & \cdots & -\delta_1 fk^{-1}n_L \\ -\delta_2 fk^{-1}n_1 & -\delta_2(fk^{-1}n_2+1) & \cdots & -\delta_1 fk^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_2 Fk^{-1}n_1 & -\delta_1 Fk^{-1}n_2 & \cdots & -\delta_I(Fk^{-1}n_L+1) \end{pmatrix} \tag{56}$$

$$= \begin{pmatrix} \delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S \end{pmatrix} \begin{pmatrix} -(fk^{-1}n_1+1) & -fk^{-1}n_2 & \cdots & -fk^{-1}n_L \\ -fk^{-1}n_1 & -(fk^{-1}n_2+1) & \cdots & -fk^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -fk^{-1}n_1 & -fk^{-1}n_2 & \cdots & -(fk^{-1}n_L+1) \end{pmatrix}$$

We can apply Corollary 5, since the matrices A and F_1 in (51) and (52) reduce to

$$\begin{pmatrix} -(k^{-1}n_1+1) & -k^{-1}n_2 & \cdots & -k^{-1}n_L \\ -k^{-1}n_1 & -(k^{-1}n_2+1) & \cdots & -k^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -k^{-1}n_1 & -k^{-1}n_2 & \cdots & -(k^{-1}n_L+1) \end{pmatrix}, \tag{57}$$

$$\begin{pmatrix} -(fk^{-1}n_1+1) & -fk^{-1}n_2 & \cdots & -fk^{-1}n_L \\ -fk^{-1}n_1 & -(fk^{-1}n_2+1) & \cdots & -fk^{-1}n_L \\ \vdots & \vdots & \ddots & \vdots \\ -fk^{-1}n_1 & -fk^{-1}n_2 & \cdots & -(fk^{-1}n_L+1) \end{pmatrix}. \tag{58}$$

Both of these matrices clearly have negative diagonals. Moreover, for column i of, say, the latter matrix (58) we compute the expression

$$|F_{1,ii}| - \sum_{j\neq i} \kappa \, |F_{1,ji}| = \left(\left|fk^{-1}n_i+1\right| - (L-1)\kappa \, |f| \, k^{-1}n_i\right) > 0$$

for some $\kappa > 0$ sufficiently small, which shows that matrices A and F_1 for the market model are quasi-dominant diagonal. (For the first matrix (57) set $f = 1$ in this argument.) The matrices are, therefore, totally stable and consequently $D-$stable.[24]

The same argument applies in the case of RLS and SG learning by the different types of agents. Thus we can state:

Proposition 9 *The symmetric equilibrium of the market model (8) is globally stable under learning*
(i) when the agents use RLS learning with differential gains (i.e. algorithm (36)-(39) under Condition 2);
or
(ii) when some suppliers use RLS and other suppliers use the SG algorithm.

[24] See e.g. pp.165-168 of (Quirk and Saposnik 1968) for these auxiliary concepts and results.

These results show that stability of the symmetric REE in the market model under the assumption of homogenous forecasts and learning rules is not at all restrictive. This model continues to be stable in the presence of the use of heterogenous learning rules by different heterogenous suppliers of the good.[25]

8 Case with Rational and Learning Agents

We continue with the class of models in Section 2 with two types of agents. However, we now assume that one type of agent has rational expectations (RE) while the other type of agent is learning. Such situations have occasionally been considered in the previous literature, see for example (Sargent 1999), (Cho, Williams, and Sargent 2001) and (Carlstrom and Fuerst 2001).[26] We now provide general conditions for stability in this case and then apply them to the model of monetary policy in Example 2.

Consider the class of models (1)-(2) and assume now that agent of type 1 is learning via RLS, while agent of type 2 has RE at every point of time (even outside the equilibrium). Obviously, the MSV solutions continue to take the same form as before and we examine stability of this class of solutions. Assume that the agent of type 1 has the PLM and corresponding forecast

$$\begin{aligned} y_t &= a_1 + b_1 w_t = \varphi_1' z_t, \\ \hat{E}_t^1 y_{t+1} &= a_1 + b_1 F w_t. \end{aligned} \tag{59}$$

Agent 2 has RE and knows that agent 1 is learning and the influence of the learning on the actual outcome of the economy. He makes use of this knowledge in forming his own forecasts. Given the forecast of agent 1, the ALM of the economy is

$$y_t = \alpha + A_1 a_1 + A_1 b_1 F w_t + A_2 E_t^2 y_{t+1} + B w_t,$$

where we no longer use the "^" symbol for agent 2's forecast since he has RE. Agent 2 knows the above ALM and makes use of this to form his own forecast.

Guessing that the MSV solution for y_t takes the same form as before, his forecast is

$$E_t^2 y_{t+1} = a_2 + b_2 F w_t \tag{60}$$

Plugging this into the ALM yields

$$y_t = \alpha + A_1 a_1 + A_2 a_2 + (A_1 b_1 F + A_2 b_2 F + B) w_t$$

[25] We remark that the (Lucas 1973) Aggregate Supply model, discussed e.g. in Chapter 2 of (Evans and Honkapohja 2001), can also be shown to be stable when different suppliers use heterogenous learning rules.

[26] (Evans, Honkapohja, and Sargent 1993) analysed the structure of equilibria with rational and boundedly rational agents in the standard overlapping generations model. See (Bomfim 2001) for references on other models of economies with heterogenous agents in terms of sophistication in forecasting.

The rational forecast for agent 2, given this ALM is,

$$E_t^2 y_{t+1} = \alpha + A_1 a_1 + A_2 a_2 + (A_1 b_1 F + A_2 b_2 F + B) F w_t \tag{61}$$

Given a_1, and b_1, we require agent 2 to have always RE at every point of time. This will be so if the coefficients in (60) and (61) are equal, i.e. if

$$\begin{aligned} a_2 &= \alpha + A_1 a_1 + A_2 a_2, & (62)\\ b_2 &= A_1 b_1 F + A_2 b_2 F + B. & (63)\end{aligned}$$

One can then solve for a_2 and b_2 from (62)-(63) after vectorizing the latter equation. This yields

$$\begin{aligned} a_2 &= (I - A_2)^{-1}(\alpha + A_1 a_1), & (64)\\ vecb_2 &= (I - F' \otimes A_2)^{-1}(F' \otimes A_1 vecb_1 + vecB). & (65)\end{aligned}$$

Note that (64)-(65) determine the RE values of a_2 and b_2 as functions of a_1 and b_1. Solving (64)-(65) with $a_1 = a_2$ and $vecb_1 = vecb_2$ just gives the symmetric RE value for a_1, b_1.

The right-hand sides of (64)-(65) lead to the $T-$mapping

$$\begin{aligned} a_1 &\to A_2(I - A_2)^{-1}(\alpha + A_1 a_1) + \alpha + A_1 a_1,\\ vecb_1 &\to (F' \otimes A_2)(I - F' \otimes A_2)^{-1}(F' \otimes A_1 vecb_1 + vecB) + (F' \otimes A_1)vecb_1 + vecB.\end{aligned}$$

For E-stability, we proceed as before. Given the PLM of agent 1, stability of learning dynamics is governed by the ODE for a_1, i.e.

$$\dot{a}_1 = [A_1 + A_2(I - A_2)^{-1}A_1 - I]a_1 + \alpha + A_2(I - A_2)^{-1}\alpha \tag{66}$$

and that for b_1 given by

$$\begin{aligned} vec\dot{b}_1 = {} & [F' \otimes A_1 + (F' \otimes A_2)(I - F' \otimes A_2)^{-1}(F' \otimes A_1) - I]vecb_1 + \\ & [(F' \otimes A_2)(I - F' \otimes A_2)^{-1} + I]vecB. \end{aligned} \tag{67}$$

The symmetric equilibrium will be locally stable under learning iff the differential equations (66)-(67) are locally asymptotically stable at the point, which proves the following proposition:

Proposition 10 *Consider the class of models (1)-(2), where agent 1 uses RLS learning and agent 2 has RE. The symmetric equilibrium of this model is stable under learning iff the eigenvalues of the matrices*

$$\begin{gathered} A_1 + A_2(I - A_2)^{-1}A_1 - I,\\ F' \otimes A_1 + (F' \otimes A_2)(I - F' \otimes A_2)^{-1}(F' \otimes A_1) - I, \end{gathered}$$

have negative real parts.

8.1 Application to Monetary Policy

We now apply Proposition 10 to our Example 2 on monetary policy. Appending the interest rule (12) to equations (9) and (10), the reduced form of the model takes the form

$$\begin{aligned}\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} &= \begin{pmatrix} -\phi \\ -\lambda\phi \end{pmatrix}\chi_0 + \begin{pmatrix} 1 & \phi \\ \lambda & \beta+\lambda\phi \end{pmatrix}\begin{pmatrix} \hat{E}_t^P z_{t+1} \\ \hat{E}_t^P \pi_{t+1} \end{pmatrix} + \\ & \begin{pmatrix} -\phi\chi_z & -\phi\chi_\pi \\ -\lambda\phi\chi_z & -\lambda\phi\chi_\pi \end{pmatrix}\begin{pmatrix} \hat{E}_t^{CB} z_{t+1} \\ \hat{E}_t^{CB} \pi_{t+1} \end{pmatrix} + \\ & \begin{pmatrix} -\phi\chi_u & 1-\phi\chi_g \\ 1-\lambda\phi\chi_u & \lambda(1-\chi_g) \end{pmatrix}\begin{pmatrix} u_t \\ g_t \end{pmatrix}. \end{aligned} \tag{68}$$

For future reference, we write the above system in a general form

$$\begin{aligned} y_t &= \alpha + A^P \hat{E}_t^P y_{t+1} + A^{CB}\hat{E}_t^{CB} y_{t+1} + Bw_t, \\ w_t &= Fw_{t-1} + v_t. \end{aligned} \tag{69}$$

where $y_t = (z_t, \pi_t)'$, $w_t = (u_t, g_t)'$ and A^P, A^{CB}, B denote the right hand matrices in (68), and F is the (diagonal) matrix appearing in (11), namely

$$F = \begin{pmatrix} \rho & 0 \\ 0 & \mu \end{pmatrix}. \tag{70}$$

We assume that $\chi_z \geq 0$ and $\chi_\pi \geq 0$ in (12).

8.1.1 Central Bank Has RE While Private Sector is Learning

When the central bank has RE and the private sector is learning, the two matrices in Proposition 10 reduce to

$$\begin{aligned} & A^P + A^{CB}(I - A^{CB})^{-1}A^P - I \\ &= (1+\varphi\chi_z + \lambda\varphi\chi_\pi)^{-1}\begin{pmatrix} -\phi(\chi_z+\lambda\chi_\pi) & \phi(1-\beta\chi_\pi) \\ \lambda & -[(1-\beta)(1+\phi\chi_z)+\lambda\phi(\chi_\pi-1)] \end{pmatrix} \end{aligned} \tag{71}$$

and

$$F' \otimes A^P + (F' \otimes A^{CB})(I - F' \otimes A^{CB})^{-1}(F' \otimes A^P) - I = \begin{pmatrix} B_\rho & 0 \\ 0 & B_\mu \end{pmatrix} \tag{72}$$

where

$$\begin{aligned} B_\rho &= [1+\rho(\phi\chi_z+\lambda\phi\chi_\pi)]^{-1} \\ & \begin{pmatrix} -[1-\rho+\rho\phi(\chi_z+\lambda\chi_\pi)] & \rho\phi(1-\beta\rho\chi_\pi) \\ \lambda\rho & -[1-\beta\rho+(1-\beta\rho)\rho\phi\chi_z+\lambda\rho\phi(\chi_\pi-1)] \end{pmatrix}, \end{aligned} \tag{73}$$

and B_μ takes the same form as B_ρ with μ replacing ρ. The trace and determinant of (71) are, respectively,

$$-(1+\phi\chi_z+\lambda\phi\chi_\pi)^{-1}[1-\beta+(2-\beta)\phi\chi_z+\lambda\phi(2\chi_\pi-1)],$$
$$\phi(1+\phi\chi_z+\lambda\phi\chi_\pi)^{-1}[(1-\beta)\chi_z+\lambda(\chi_\pi-1)].$$

It is easy to check that the determinant is positive iff $(1-\beta)\chi_z+\lambda(\chi_\pi-1)>0$ and this also suffices to make the trace negative. As for the matrix (72), we note that it is block diagonal so that its eigenvalues are those of B_ρ and B_μ and since the latter two matrices are, respectively, symmetric in ρ,μ it suffices to look only at B_ρ for stability. The trace and determinant of (73) are respectively

$$-[1+\rho(\phi\chi_z+\lambda\chi_\pi)]^{-1}[2-\rho(1+\beta)+\rho\phi\{(2-\beta\rho)\chi_z+\lambda(2\chi_\pi-1)\}],$$
$$[1+\rho(\phi\chi_z+\lambda\phi\chi_\pi)]^{-1}[(1-\rho)(1-\beta\rho)+\rho\phi\{(1-\beta\rho)\chi_z+\lambda(\chi_\pi-1)\}].$$

It is easy to check that $(1-\beta)\chi_z+\lambda(\chi_\pi-1)>0$ implies that the trace above is negative and determinant positive for (73). This proves the following corollary.

Corollary 11 *Assume that for the model (69), the private sector is learning via RLS while the central bank always has RE. The dynamics of the economy is then locally stable if and only if* $(1-\beta)\chi_z+\lambda(\chi_\pi-1)>0$.

Condition $(1-\beta)\chi_z+\lambda(\chi_\pi-1)>0$ is precisely the Taylor principle that characterized learnability in (Bullard and Mitra 2001), where both the central bank and the private sector were learning via RLS with identical learning rules. The same principle determines stability also when only the private sector is learning as above. Since the central bank has now so much more information than the private sector, it is able to neutralize the destabilizing influence of the latter (which arises since A^P has an eigenvalue more than 1) by subscribing to the Taylor principle in its interest rule. See (Honkapohja and Mitra 2002) for more on the intuition behind the stabilizing influence from the central bank and the de-stabilizing effect arising from the behavior of the private agents.

8.1.2 Central Bank is Learning While Private Sector Has RE

Consider now the situation when the central bank is learning while the private agents always have RE in the sense defined above.[27] In this case we have

$$A^{CB}+A^P(I-A^P)^{-1}A^{CB}-I=\begin{pmatrix}\lambda^{-1}(1-\beta)\chi_z-1 & \lambda^{-1}(1-\beta)\chi_\pi\\ \chi_z & \chi_\pi-1\end{pmatrix}$$

The determinant and trace of the above matrix equal, respectively,

$$-\lambda^{-1}[(1-\beta)\chi_z+\lambda(\chi_\pi-1)],$$
$$\lambda^{-1}[(1-\beta)\chi_z+\lambda(\chi_\pi-1)]-1.$$

[27] This is the situation which is sometimes assumed in the literature, for instance, in (Sargent 1999), (Cho, Williams, and Sargent 2001) and (Carlstrom and Fuerst 2001).

The determinant is positive if and only if $(1-\beta)\chi_z + \lambda(\chi_\pi - 1) < 0$ and this also makes the trace negative. Therefore, a necessary condition for the equilibrium to be stable is that the Taylor principle be *violated*.

As before, it can be shown that the matrix corresponding to (72) (after inter-changing the roles of A^{CB} and A^P there) is block diagonal with the diagonal matrices being symmetric in μ and ρ. The eigenvalues of the diagonal matrix corresponding to μ are given by -1 and

$$-[(1-\mu)(1-\beta\mu) - \mu\lambda\phi]^{-1}[(1-\mu)(1-\beta\mu) + \mu\phi\{(1-\beta\mu)\chi_z + \lambda(\chi_\pi - 1)\}] > 0$$

provided $(1-\mu)(1-\beta\mu) - \mu\lambda\phi \neq 0$. This enables us to prove the following corollary.

Corollary 12 *Assume that for the model (69), the private sector always has RE while the central bank is learning via RLS. The necessary and sufficient conditions for the symmetric equilibrium to be locally stable are*

$$\begin{aligned}(1-\beta)\chi_z + \lambda(\chi_\pi - 1) &< 0,\\ [(1-\mu)(1-\beta\mu) - \mu\lambda\phi]^{-1}[(1-\mu)(1-\beta\mu) + \mu\phi\{(1-\beta\mu)\chi_z + \lambda(\chi_\pi - 1)\}] &> 0,\\ [(1-\rho)(1-\beta\rho) - \rho\lambda\phi]^{-1}[(1-\rho)(1-\beta\rho) + \rho\phi\{(1-\beta\rho)\chi_z + \lambda(\chi_\pi - 1)\}] &> 0.\end{aligned}$$

The result in (Bullard and Mitra 2001) and in the previous section has now been turned on its head by this extreme assumption of rationality of the private sector *vis-a-vis* the central bank. We note that, in general, violation of the Taylor principle is not sufficient for stability of the equilibrium. This is because the latter two conditions in Corollary 12 depend also on μ and ρ. In fact it can be checked numerically for plausible values of parameters used in (Woodford 1999) that equilibrium may be either stable or unstable even when $(1-\beta)\chi_z + \lambda(\chi_\pi - 1) < 0$.

A case of stability arises when the policy does not react at all to forecasts, i.e. $\chi_z = \chi_\pi = 0$. This is natural, since by assumption the private economy has already converged to the MSV REE and so the choice of the interest rate instrument rule need not then be based on considerations of stability under learning. However, we note that interest rate rules that react only to exogenous observables are problematic, as they lead to indeterminacy (and also instability under learning if in fact private agents do not have RE).

9 Lagged Endogenous Variables

9.1 General Analysis

The analysis discussed in the previous sections can be extended to cover models with lagged endogenous variables. We develop this only briefly since most of the formal analysis in the previous sections goes through with only minor changes. There is one important difference though: the sense of convergence is only local also for linear frameworks. This is evident already from the observation that in most cases there are two MSV solutions to the model, so that neither of them can be globally stable under learning.

Consider the class of models

$$\begin{aligned} y_t &= \alpha + A_1\hat{E}_t^1 y_{t+1} + A_2\hat{E}_t^2 y_{t+1} + Dy_{t-1} + Bw_t, \\ w_t &= Fw_{t-1} + \varepsilon_t. \end{aligned} \tag{74}$$

The MSV solutions are now of the form

$$y_t = a + bw_t + cy_{t-1}$$

and in general there are two (or zero) such solutions. It should be noted that such a solution may or may not be stationary and we will keep track of this issue.

Agents forecast using the PLMs

$$\begin{aligned} y_t &= a_1 + b_1 w_t + c_1 y_{t-1} = \varphi_1' z_t \\ y_t &= a_2 + b_2 w_t + c_2 y_{t-1} = \varphi_2' z_t \end{aligned}$$

where $z_t = (1, w_t', y_{t-1}')'$ and $\varphi_i' = (a_i, b_i, c_i)$ for $i = 1, 2$ in this section. (We have kept the same general notation z_t and φ_i' for the state variable and parameters. This should not cause any confusion.) The corresponding forecast functions are

$$\begin{aligned} \hat{E}_t^i y_{t+1} &= a_i + b_i F w_t + c_i \hat{E}_t^i y_t \\ &= a_i + c_i a_i + c_i^2 y_{t-1} + (b_i F + c_i b_i) w_t, \end{aligned} \tag{75}$$

where we have assumed that the contemporaneous y_t is not available in the information set of the agents. (This assumption is often used in the literature.)

Inserting the forecasts (75) into the model (74), one obtains the ALM

$$\begin{aligned} y_t &= \alpha + A_1(a_1 + c_1 a_1) + A_2(a_2 + c_2 a_2) + (A_1 c_1^2 + A_2 c_2^2 + D) y_{t-1} + \\ &\quad [A_1(b_1 F + c_1 b_1) + A_2(b_2 F + c_2 b_2) + B] w_t \\ &\equiv T(\varphi_1', \varphi_2') z_t, \end{aligned}$$

where the T map is now given by

$$\begin{aligned} a_i &\to \alpha + A_1(a_1 + c_1 a_1) + A_2(a_2 + c_2 a_2), \\ b_i &\to A_1(b_1 F + c_1 b_1) + A_2(b_2 F + c_2 b_2) + B, \\ c_i &\to A_1 c_1^2 + A_2 c_2^2 + D. \end{aligned}$$

We assume that agents use versions of RLS and their algorithms continue to be given by (36)-(39) and the gain sequences furthermore satisfy Condition 2. As before, local stability of learning dynamics is governed by the associated ordinary differential equation

$$\begin{aligned} d\varphi_1/d\tau &= \delta_1 S_1^{-1} M_z(\varphi_1, \varphi_2)(T(\varphi_1', \varphi_2')' - \varphi_1), \\ dS_1/d\tau &= \delta_1(M_z(\varphi_1, \varphi_2) - S_1), \\ d\varphi_2/d\tau &= \delta_2 S_2^{-1} M_z(\varphi_1, \varphi_2)(T(\varphi_1', \varphi_2')' - \varphi_2), \\ dS_2/d\tau &= \delta_2(M_z(\varphi_1, \varphi_2) - S_2). \end{aligned}$$

The only difference from Section 5 is that the moment matrix $M_z(\varphi_1, \varphi_2)$ now depends on φ_1 and φ_2. Nevertheless, it can be shown that local stability is finally governed by the smaller system

$$\begin{aligned} d\varphi_1/d\tau &= \delta_1(T(\varphi_1', \varphi_2')' - \varphi_1), \\ d\varphi_2/d\tau &= \delta_2(T(\varphi_1', \varphi_2')' - \varphi_2). \end{aligned}$$

When $\delta_1 = \delta_2 = 1$ as in standard RLS, stability would be governed solely by the E-stability equations. In general, however, δ_1 and δ_2 affect stability.

One can also consider the case when agent 1 uses RLS and agent 2 the SG algorithm in their estimation, i.e., the learning algorithms are given by (46), (47), and (48). Replicating the arguments in Section 6, one can show that local stability is governed by the following system

$$\begin{aligned} d\varphi_1/d\tau &= (T(\varphi_1', \varphi_2')' - \varphi_1), \\ d\varphi_2/d\tau &= M_z(\varphi_1, \varphi_2)(T(\varphi_1', \varphi_2')' - \varphi_2). \end{aligned}$$

9.2 Application to Cagan Model

We now apply the preceding considerations to Example 3 of Section 2.1, i.e. the Cagan model (13). Assume that the PLM of agent $i = 1, 2$ is of the form

$$p_t = a_i + b_i v_t + c_i p_{t-1}$$

with the corresponding forecast

$$\begin{aligned} \hat{E}_t^i p_{t+1} &= a_i + c_i \hat{E}_t^i p_t = a_i + c_i(a_i + c_i p_{t-1} + b_i v_t) \\ &= a_i + c_i a_i + c_i^2 p_{t-1} + c_i b_i v_t. \end{aligned}$$

(We have again assumed that agents do not use the contemporaneous price to make their forecasts.) Plugging the forecasts above into (13) yields the ALM of p_t

$$\begin{aligned} p_t &= \alpha + \omega p_{t-1} + \sum_{i=1}^{2} \eta_i(a_i + c_i a_i + c_i^2 p_{t-1} + c_i b_i v_t) + v_t \\ &= \alpha + \eta_1 a_1(1 + c_1) + \eta_2 a_2(1 + c_2) + (\eta_1 c_1^2 + \eta_2 c_2^2 + \omega)p_{t-1} \\ &\quad +(1 + \eta_1 c_1 b_1 + \eta_2 c_2 b_2)v_t \\ &\equiv T(\varphi_1', \varphi_2')z_t, \end{aligned}$$

where $z_t = (1, v_t, p_{t-1})'$ and $\varphi_i' = (a_i, b_i, c_i)$ for $i = 1, 2$.

The map from the PLM to the ALM, i.e., the T map is

$$\begin{aligned} T(\varphi_1', \varphi_2')' &= [\alpha + \eta_1 a_1(1 + c_1) + \eta_2 a_2(1 + c_2), \eta_1 c_1^2 + \eta_2 c_2^2 + \omega, \\ &\quad 1 + \eta_1 c_1 b_1 + \eta_2 c_2 b_2] \end{aligned}$$

and the symmetric equilibria are

$$\begin{aligned}\bar{a}_1 &= \bar{a}_2 \equiv \bar{a} = 0,\\ \bar{b}_1 &= \bar{b}_2 \equiv \bar{b} = [1 - \bar{c}(\eta_1 + \eta_2)]^{-1},\\ \bar{c}_1 &= \bar{c}_2 \equiv \bar{c} = [2(\eta_1 + \eta_2)]^{-1}[1 \pm \sqrt{1 - 4(\eta_1 + \eta_2)\omega}].\end{aligned}$$

Note that there are usually two such solutions, as exemplified by the expression for $\bar{c}$ and we denote them by $\bar{c}_+$ and $\bar{c}_-$ for the time being. Correspondingly, we use the notation $\bar{b}_+$ and $\bar{b}_-$.

We first observe that, given our assumptions on the structural parameters, $\bar{c}_-$ is the uniquely stationary solution. This follows as a special case for the same class of models considered in (Evans and Honkapohja 2001), p. 203, since

$$|\eta_1 + \eta_2 + \omega| = \left|\frac{d + g_1 + g_2}{1 + g_1 + g_2}\right| < 1.$$

It can be shown that the $\bar{c}_-$ solution is always E-stable (and hence stable under RLS learning) in the case of homogenous learning rules.[28] For this point onwards we refer to the E-stable REE as $(0, \bar{b}, \bar{c})$ without the subscript "–" to simplify the notation. In our case the condition for stability is given by (see (Evans and Honkapohja 2001), p. 204)

$$-\sqrt{1 - 4(\eta_1 + \eta_2)\omega} < 1 - 2(\eta_1 + \eta_2)$$

which reduces to

$$-\sqrt{(1 + g_1 + g_2)^2 - 4d(g_1 + g_2)} < 1 - (g_1 + g_2) \tag{76}$$

If $g_1 + g_2 \leq 1$, (76) is automatically satisfied. If $g_1 + g_2 > 1$, (76) is equivalent to (after squaring both sides and simplifying)

$$4(1 - d)(g_1 + g_2) > 0$$

which is obviously true since $|d| < 1$. Thus we have:

Proposition 13 *The uniquely stationary, symmetric equilibrium in the Cagan model is E-stable.*

In the next section we check whether this symmetric equilibrium continues to be locally stable under heterogenous learning rules.

[28] The $\bar{b}_+$ solution is always unstable, see (Evans and Honkapohja 2001) p. 204.

9.2.1 RLS and SG Learning in Cagan Model

We now analyze heterogenous learning in the Cagan model when agent 1 uses RLS learning and agent 2 uses the SG algorithm as follows

$$\begin{aligned}\varphi_{1,t} &= \varphi_{1,t-1} + \gamma_t R_t^{-1} z_{t-1}(p_{t-1} - \varphi'_{1,t-1} z_{t-1})', \\ R_t &= R_{t-1} + \gamma_t (z_{t-1} z'_{t-1} - R_{t-1}), \\ \varphi_{2,t} &= \varphi_{2,t-1} + \gamma_t z_{t-1}(p_{t-1} - \varphi'_{2,t-1} z_{t-1})',\end{aligned}$$

where $T(\varphi'_1, \varphi'_2) z_t$. Now $M_z(\varphi_1, \varphi_2) \equiv \lim_t E z_{t-1} z'_{t-1} = \lim_t E \begin{pmatrix} 1 \\ p_{t-2} \\ v_{t-1} \end{pmatrix} \begin{pmatrix} 1 & p_{t-2} & v_{t-1} \end{pmatrix}$,

so that

$$M_z(\varphi_1, \varphi_2) = \lim_t E \begin{pmatrix} 1 & p_{t-2} & v_{t-1} \\ p_{t-2} & p_{t-2}^2 & p_{t-2} v_{t-1} \\ v_{t-1} & p_{t-2} v_{t-1} & v_{t-1}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_p^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{pmatrix}$$

where

$$\sigma_p^2 = [1 - (\eta_1 c_1^2 + \eta_2 c_2^2 + \omega)^2]^{-1} \sigma_v^2. \tag{77}$$

As was shown in Section 9, local stability is governed by the system

$$\begin{aligned} d\varphi_1/d\tau &= T(\varphi'_1, \varphi'_2)' - \varphi_1 \\ d\varphi_2/d\tau &= M_z(\varphi_1, \varphi_2)(T(\varphi'_1, \varphi'_2)' - \varphi_2). \end{aligned}$$

Simplifying this, one can show that the symmetric equilibrium $(0, \bar{b}, \bar{c})$ is locally stable provided the eigenvalues of the following 6×6 matrix

$$\begin{pmatrix} \eta_1(1+\bar{c}) - 1 & 0 & 0 & \eta_2(1+\bar{c}) & 0 & 0 \\ 0 & 2\eta_1\bar{c} - 1 & 0 & 0 & 2\eta_2\bar{c} & 0 \\ 0 & \eta_1\bar{b} & \eta_1\bar{c} - 1 & 0 & \eta_2\bar{b} & \eta_2\bar{c} \\ \eta_1(1+\bar{c}) & 0 & 0 & \eta_2(1+\bar{c}) - 1 & 0 & 0 \\ 0 & 2\eta_1\bar{c}\sigma_p^2 & 0 & 0 & (2\eta_2\bar{c} - 1)\sigma_p^2 & 0 \\ 0 & \eta_1\bar{b}\sigma_v^2 & \sigma_v^2\eta_1\bar{c} & 0 & \eta_2\bar{b}\sigma_v^2 & (\eta_2\bar{c} - 1)\sigma_v^2 \end{pmatrix} \tag{78}$$

have negative real parts. Appendix A.2 proves the following proposition.

Proposition 14 *The unique, stationary, symmetric equilibrium of the Cagan model continues to be locally stable when one agent uses RLS and the other the SG algorithm in their learning rules.*

10 Concluding Remarks

Most macroeconomic models are based on the assumption of structural homogeneity, i.e. of the representative agent, and in the literature on learning this assumption is usually extended to include the learning rules of the agents. In this paper we have considered the significance of this assumption for stability of learning dynamics by studying the implications of structural heterogeneity, which is captured by the differential effect of the expectations of the different agents on the economy. The class of models we consider includes forward looking models with or without lags. Several cases of structural and expectational heterogeneity were analyzed.

We started by showing that introducing heterogeneity only in beliefs but not in learning rules has no significant consequences, as the convergence conditions are the same as in the corresponding model with homogenous expectations. This result was then extended by analyzing the implications of heterogeneity in learning rules (and not only forecasts) when agents are boundedly rational and are learning about key parameters of the economy. We also briefly considered the case, where some agents have RE continuously while other agents are learning.

In general, the stability conditions for learning are affected by this kind of heterogeneity, but this is not always the case. Some standard models, which have been found to converge to REE under homogenous expectations and learning, continue to do so in the presence of heterogenous expectations and learning rules. This shows that the assumption of homogenous expectations and learning rules is not always as restrictive as it may seem at first sight.

There are, of course, models for which heterogenous learning affects the conditions for convergence of learning. An important case is the basic forward looking model of monetary policy commonly considered in the New Keynesian literature. In this paper we considered this model for the case in which one class of agents has RE while the other is learning. The companion paper (Honkapohja and Mitra 2002) provides a thorough analysis of this model and examines to what extent heterogeneity can affect the desirability of different interest rate rules advocated in the literature.

The analysis and the results in this paper are based on the assumption of symmetric information, so that agents observe and make forecasts on the same set of "macro" variables in the economy. This setting is natural in many models, but extensions to our analysis are going to be needed for some specific settings. For example, we have not considered the learnability of non-MSV REE. Perhaps more importantly, we stress that adaptive learning in economies with asymmetric information should be considered further as the existing literature is far from comprehensive.

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A Appendix

A.1 Convergence of the Learning Algorithms

We now develop the technical details that were also used in the previous sections. We will work out the details in the case where one agents uses RLS and the other SG learning and then indicate the necessary modifications for the case of different gain sequences.

Details for Theorem 6: We begin by rewriting (46), (47), and (48) as a stochastic recursive algorithm after making a timing change in (46) and (47) by defining $S_{t-1} = R_t$.[29] These algorithms start from the general form

$$\theta_t = \theta_{t-1} + \gamma_t H(\theta_{t-1}, X_t) + \gamma_t^2 \rho_t(\theta_{t-1}, X_t) \tag{79}$$

where θ_t is a vector of parameter estimates and X_t is the state vector. In our case we have $\theta_t' = (\varphi_{1,t}', \varphi_{2,t}', vec(S_t))$ and $X_t' = (1, w_t', w_{t-1}')$.

Since the $T-$map continues to be given by (18), we substitute (35) into (46) and get

$$\begin{aligned} \varphi_{1,t} &= \varphi_{1,t-1} + \gamma_t S_{t-1}^{-1} z_{t-1}(T(\varphi_{1,t-1}', \varphi_{2,t-1}')z_{t-1} - \varphi_{1,t-1}' z_{t-1})' \\ &\quad + (\gamma_{1,t} - \gamma_t) S_{t-1}^{-1} z_{t-1}(T(\varphi_{1,t-1}', \varphi_{2,t-1}')z_{t-1} - \varphi_{1,t-1}' z_{t-1})'. \end{aligned}$$

This gives us the φ_1 components of the function $H(\theta_{t-1}, X_t)$ in (79), which we denote by $H_1(z_{t-1}, \varphi_{1,t-1}, \varphi_{2,t-1}, S_{t-1})$. In other words,

$$H_1(z_{t-1}, \varphi_{1,t-1}, \varphi_{2,t-1}, S_{t-1}) = S_{t-1}^{-1} z_{t-1} z_{t-1}'(T(\varphi_{1,t-1}', \varphi_{2,t-1}')' - \varphi_{1,t-1}). \tag{80}$$

Regarding the second order in γ_t term in (79), we have

$$\rho_{\varphi,t}(\theta_{t-1}, X_t) = \frac{\gamma_{1,t} - \gamma_t}{\gamma_t^2} S_{t-1}^{-1} z_{t-1}(T(\varphi_{1,t-1}', \varphi_{2,t-1}')z_{t-1} - \varphi_{1,t-1}' z_{t-1})',$$

and the validity of the method requires that this be bounded in t. This is easily established as by Assumption A (with $K_i \leq 1$ without loss of generality) we have $\gamma_{1,t}/\gamma_t \leq 1$ $\Rightarrow \gamma_{1,t}/\gamma_t \leq 1 + K\gamma_t$ for any $K > 0 \Rightarrow \frac{\gamma_{1,t} - \gamma_t}{\gamma_t^2} \leq 1$.

For (47) we can write

$$\begin{aligned} S_t &= S_{t-1} + \gamma_t(z_t z_t' - S_{t-1}) + (\gamma_{1,t+1} - \gamma_t)(z_t z_t' - S_{t-1}) \\ &= S_{t-1} + \gamma_t(z_t z_t' - S_{t-1}) + \gamma_t^2 (\frac{\gamma_{1,t+1} - \gamma_t}{\gamma_t^2})(z_t z_t' - S_{t-1}). \end{aligned}$$

Thus the S components of the function $H(\theta_{t-1}, X_t)$ are given by

$$H_S(z_t, S_{t-1}) \equiv z_t z_t' - S_{t-1} \tag{81}$$

[29] See Chapters 7 and 8 of (Evans and Honkapohja 2001) for an exposition of the technique.

while the second order in γ_t term

$$\rho_{S,t}(\theta_{t-1}, X_t) = (\frac{\gamma_{1,t+1} - \gamma_t}{\gamma_t^2})(z_t z_t' - S_{t-1})$$

is bounded in t since

$$\frac{\gamma_{1,t+1} - \gamma_t}{\gamma_t^2} = \frac{\gamma_{1,t+1}}{\gamma_{t+1}} \left(\frac{\gamma_{t+1}}{\gamma_t}\right)^2 \frac{1}{\gamma_{t+1}} - \frac{1}{\gamma_t} \leq \frac{1}{\gamma_{t+1}} - \frac{1}{\gamma_t}$$

by Assumption A.

Finally, in a similar manner we get the φ_2 components of the function $H(\theta_{t-1}, X_t)$ in (48), which for future use we denote by $H_2(t, z_{t-1}, \varphi_{1,t-1}, \varphi_{2,t-1}, S_{t-1})$.

Now

$$\lim_{t\to\infty} EH_1(z_{t-1}, \varphi_1, \varphi_2, S) = S^{-1}M_z(T(\varphi_1', \varphi_2')' - \varphi_1).$$

where M_z is defined in (41). Similarly

$$\lim_{t\to\infty} EH_S(z_{t-1}, S, t) = M_z - S.$$

and

$$\lim_{t\to\infty} EH_2(z_{t-1}, \varphi_1, \varphi_2, S, t) = M_z(T(\varphi_1', \varphi_2')' - \varphi_2).$$

The associated differential equation is then defined by

$$d\theta/d\tau = h(\theta) = \lim{}_{t\to\infty} EH(t, \theta_{t-1}, X_t)$$

and in our case it boils down to

$$\begin{aligned} d\varphi_1/d\tau &= S^{-1}M_z(T(\varphi_1', \varphi_2')' - \varphi_1), \\ dS/d\tau &= M_z - S, \\ d\varphi_2/d\tau &= M_z(T(\varphi_1', \varphi_2')' - \varphi_2). \end{aligned}$$

Since the second set of equations is globally stable with $S \to M_z$ from any starting point, stability is determined entirely by the smaller dimensional system

$$\begin{aligned} d\varphi_1/d\tau &= (T(\varphi_1', \varphi_2')' - \varphi_1), \\ d\varphi_2/d\tau &= M_z(T(\varphi_1', \varphi_2')' - \varphi_2). \end{aligned}$$

The rest of the proof is given in the main text, from equations (49).

Details for Theorem 4: In this case, one proceeds for both agents as above for agent 1, but one can write the gain sequence as $\gamma_{1,t} = \gamma_t(\xi_{i,t}\hat{\gamma}_{1,t}\gamma_t^{-1})$ and treat $\xi_{i,t}\hat{\gamma}_{1,t}\gamma_t^{-1}$ as an additional state variable that evolves exogenously from the rest of the system. With random gain sequences in (80) and (81) we get for agent 1

$$\lim_{t\to\infty} EH_1(\xi_{i,t}\gamma_{1,t}\gamma_t^{-1}, z_{t-1}, \varphi_1, \varphi_2, S) = \delta_1 S^{-1} M_z (T(\varphi_1', \varphi_2')' - \varphi_1)$$

and

$$\lim_{t\to\infty} EH_S(\xi_{i,t}\gamma_{1,t}\gamma_t^{-1}, z_{t-1}, S, t) = \delta_1 (M_z - S)$$

since $\lim_{t\to\infty} E(\gamma_{1,t}\gamma_t^{-1}) = \delta_1$. Doing the same for agent 2 we arrive at the associated ODE for this, given as (40) in the main text.

A.2 Proof of Proposition 14

We first determine the sign of $\bar{c}_-$ which depends on d. Note that the expression for $\bar{c}_-$ is given by the solution of

$$q(b) \equiv c^2 - (\eta_1 + \eta_2)^{-1} c + (\eta_1 + \eta_2)^{-1} \omega = 0$$

Consequently,

$$\begin{aligned} q(0) &= (\eta_1 + \eta_2)^{-1} \omega, \\ q(1) &= 1 + (\eta_1 + \eta_2)^{-1} (\omega - 1), \\ q(-1) &= 1 + (\eta_1 + \eta_2)^{-1} (\omega + 1). \end{aligned}$$

When $-1 < d < 0$, so that $-1 < \omega < 0$, we have $q(0) < 0$, $q(1) < 0$ and $q(-1) > 0$ which means that $-1 < \bar{c}_- < 0$. When $0 < d < 1$, $q(0) > 0$ and

$$q(1) = (\eta_1 + \eta_2)^{-1} (\eta_1 + \eta_2 + \omega - 1) < 0$$

since $\eta_1 + \eta_2 + \omega < 1$. Consequently, $0 < d < 1$ implies that $0 < \bar{c}_- < 1$.

We now move to an analysis of stability under learning when agents use heterogenous learning rules. Using Mathematica, one can show that two eigenvalues of the matrix, (78), when evaluated at any symmetric equilibrium $\bar{c}$, are given by -1 and $(1 + \bar{c})(\eta_1 + \eta_2) - 1$ and the remaining four eigenvalues are given by the following two characteristic polynomials

$$l(m) \equiv m^2 + m[1 - \eta_1 \bar{c} + \sigma_v^2 (1 - \eta_2 \bar{c})] + \sigma_v^2 [1 - \bar{c}(\eta_1 + \eta_2)] = 0, \tag{82}$$

and

$$\begin{aligned} p(m) &\equiv m^2 + m a_0 + a_1 = 0, \qquad (83) \\ a_0 &= 1 - 2\eta_1 \bar{c} + (1 - 2\eta_2 \bar{c})[1 - \{(\eta_1 + \eta_2)\bar{c}^2 + \omega\}^2]^{-1} \sigma_v^2, \\ a_1 &= [1 - 2(\eta_1 + \eta_2)\bar{c}][1 - \{(\eta_1 + \eta_2)\bar{c}^2 + \omega\}^2]^{-1} \sigma_v^2. \end{aligned}$$

Here (77) implies that $\sigma_p^2 = [1-\{(\eta_1+\eta_2)\bar{c}^2+\omega\}^2]^{-1}\sigma_v^2$ when evaluated at the symmetric equilibrium $\bar{c}$. $\sigma_p^2 > 0$ implies that $1 - \{(\eta_1 + \eta_2)\bar{c}^2 + \omega\}^2 > 0$. We first examine the polynomial (83). Both eigenvalues of $p(m)$ have negative real parts iff $a_0 > 0$ and $a_1 > 0$.

When $-1 < d < 0$, we have $-1 < \bar{c}_- < 0$ and hence $a_0 > 0$ and $a_1 > 0$ in $p(m)$, (83). For the same reason, the constant term and the coefficient of m in $l(m)$ in (82) are positive. Finally, $(1 + \bar{c}_-)(\eta_1 + \eta_2) - 1 < 0$ since $\eta_1 + \eta_2 < 1$. So all eigenvalues of the matrix, (78), have negative real parts and the symmetric equilibrium is stable when $-1 < d < 0$.

We now examine the case when $0 < d < 1$ so that $0 < \bar{c}_- < 1$. Note that

$$\begin{aligned}\bar{c}_- &= [2(\eta_1 + \eta_2)]^{-1}[1 - \sqrt{1 - 4(\eta_1 + \eta_2)\omega}] \\ &= \frac{1 + g_1 + g_2}{2(g_1 + g_2)}[1 - \sqrt{1 - \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2}}].\end{aligned}$$

Also

$$2(\eta_1 + \eta_2)\bar{c}_- = 1 - \sqrt{1 - \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2}} < 1.$$

Since $1 - 2(\eta_1 + \eta_2)\bar{c}_- > 0$, it follows that $1 - 2\eta_1\bar{c}_- > 2\eta_2\bar{c}_- > 0$ and analogously $1 - 2\eta_2\bar{c}_- > 0$. This implies $a_0 > 0$ and $a_1 > 0$ in $p(m)$ of (83).

In addition, for the same reason, the characteristic polynomial $l(m)$ has eigenvalues with negative real parts. Finally, we still need to show that the eigenvalue $(1+\bar{c}_-)(\eta_1 + \eta_2) - 1$ is negative, which is not obvious since $0 < \bar{c}_- < 1$. In terms of structural parameters, we have

$$\begin{aligned}&(1 + \bar{c}_-)(\eta_1 + \eta_2) - 1 \\ &= (\frac{g_1 + g_2}{1 + g_1 + g_2})[1 + \frac{1 + g_1 + g_2}{2(g_1 + g_2)}\{1 - \sqrt{1 - \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2}}\}] - 1 \\ &= 1 - \frac{2}{1 + g_1 + g_2} - \sqrt{1 - \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2}}.\end{aligned}$$

Consequently, $(1 + \bar{c}_-)(\eta_1 + \eta_2) - 1 < 0$ if and only if

$$\frac{g_1 + g_2 - 1}{1 + g_1 + g_2} < \sqrt{1 - \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2}} \tag{84}$$

If $g_1 + g_2 \leq 1$, inequality (84) is immediate. If $g_1 + g_2 > 1$, square both sides of (84) to get

$$(\frac{g_1 + g_2 - 1}{1 + g_1 + g_2})^2 + \frac{4d(g_1 + g_2)}{(1 + g_1 + g_2)^2} < 1. \tag{85}$$

(85) is equivalent to

$$\frac{(g_1+g_2)^2+1+2(g_1+g_2)(2d-1)}{(g_1+g_2)^2+1+2(g_1+g_2)} < 1. \tag{86}$$

It is easy to see now that the inequality (86) is satisfied iff $2d-1<1$, i.e., $d<1$, which is true by assumption. This proves the proposition.

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